IMS MATHS BOOK-01

Set – IX (i)

* Functions of Several Variables *

Introduction:–

Most measurable quantities in the real world do not depend on one single factor but on many factors. This indicates that functions of several variables are natural entities in the world of mathematics.

So far we have studied the concepts of limits, continuity, differentiability etc. for functions of a single variable.

— Now we introduce the concept of limit, continuity and differentiability of functions of several variables. Mainly we study these concepts for real valued functions of two variables which can be generalised to functions of several variables.

* Euclidean Space:

For a fixed $n \in \mathbb{N}$, let $\mathbb{R}^n$ be the set of all ordered n-tuples $x = (x_1, x_2, \ldots, x_n)$ where $x_1, x_2, \ldots, x_n \in \mathbb{R}$ are called the coordinates of x.

→ the elements of $\mathbb{R}^n$ are called points or vectors and denoted by x, y, z etc.

→ we define the addition of vectors and multiplication of a vector by real number (called scalar) as follows:

Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$; $\alpha \in \mathbb{R}$

then $x + y = (x_1+y_1, x_2+y_2, \ldots, x_n+y_n) \in \mathbb{R}^n$

and $\alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n) \in \mathbb{R}^n$

these two operations make $\mathbb{R}^n$ a vector space over the real field $\mathbb{R}$.

— The zero elements of $\mathbb{R}^n$ (some times called the origin or null vector) is the point $0 = (0, 0, \ldots, 0)$.

— we define the scalar product (or inner product) of two vectors x and y by

$$x \cdot y = \sum_{i=1}^{n} x_i y_i = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

and the norm of x by $\|x\| = (x \cdot x)^{1/2}$

$$= \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

The vector space $\mathbb{R}^n$ with the above inner product and norm is called n-dimensional Euclidean space.

— In particular, we get $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$ for $n = 1, 2, 3$ respectively we write

$x = (x_1, x_2)$ if $x \in \mathbb{R}^2$

$x = (x_1, x_2, x_3)$ if $x \in \mathbb{R}^3$.

Functions of Several Variables:

Let $f: X \to \mathbb{R}$. If $X \subset \mathbb{R}^n$ then f is called a function of n variables.

→ f is a function of several variables if $n > 1$.

→ $f: X \to \mathbb{R}$ is a function of two variables if $X \subseteq \mathbb{R}^2$.

$f: X \to \mathbb{R}$ is a function of three variables if $X \subset \mathbb{R}^3$.

* Neighbourhood of a point:

→ Spherical neighbourhood of a point:

Let $\mathbb{R}^n$ be the Euclidean space and $a \in \mathbb{R}^n$ (i.e. $a = (a_1, a_2, \ldots a_n)$).

If δ is any +ve real number, then the set $\{x \in \mathbb{R}^n / \|x - a\| < \delta\}$ is called an open sphere.

(since $\|x - a\| < \delta \Rightarrow \left[\sum_{i=1}^{n} (x_i - a_i)^2\right]^{1/2} < \delta$

$$\Rightarrow \sum_{i=1}^{n} (x_i - a_i)^2 < \delta^2$$

$$\Rightarrow (x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots + (x_n - a_n)^2 < \delta^2$$

The point 'a' is called the centre and 'δ' the radius of the sphere.

This open sphere is denoted by $S(a, \delta)$

— A closed sphere is denoted by $S[a, \delta]$ and is defined by

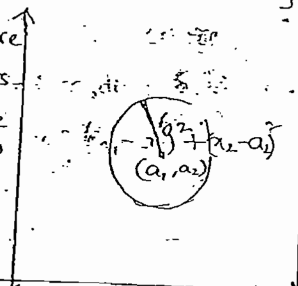
$S[a, \delta] = \{x \in \mathbb{R}^n / \|x - a\| \le \delta\}$.

— Any open sphere with 'a' as its centre is called a spherical neighbourhood of the point 'a'.

The open sphere with centre $a = (a_1, a_2) \in \mathbb{R}^2$ and radius δ is

$S(a, \delta) = \{(x_1, x_2) \in \mathbb{R}^2 / (x_1 - a_1)^2 + (x_2 - a_2)^2 < \delta^2\}$

∴ The open sphere in this case consists of all points of the cartesian plane which lie within the circle $(x_1 - a_1)^2 + (x_2 - a_2)^2 = \delta^2$.



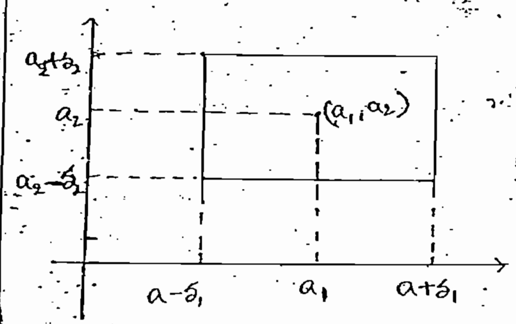
$\{(x_1, x_2) \in \mathbb{R}^2 / (x_1 - a_1)^2 + (x_2 - a_2)^2 < \delta^2\}$

* Rectangular Neighbourhood of a point:

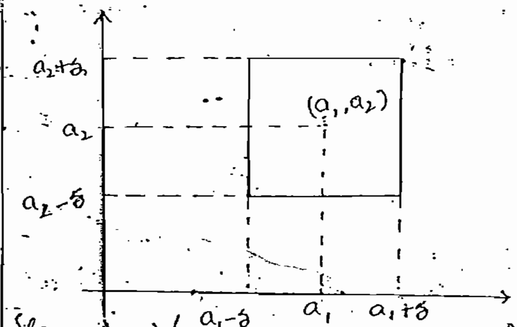
Rectangular neighbourhood of a point $a = (a_1, a_2, \ldots a_n) \in \mathbb{R}^n$ is defined to be the set $\{x \in \mathbb{R}^n / |x_i - a_i| < \delta_i\}, i = 1, 2, \ldots n$

— The rectangular neighbourhood of (a_1, a_2) is $\{(x_1, x_2) \in \mathbb{R}^2 / |x_1 - a_1| < \delta_1$ and $|x_2 - a_2| < \delta_2\}$

If in particular, $\delta_1 = \delta_2 = \delta$ then such a neighbourhood is referred to as a square neighbourhood of side 2δ.



$\{(x_1, x_2) \in \mathbb{R}^2 / |x_1 - a_1| < \delta_1, |x_2 - a_2| < \delta_2\}$



$\{(x_1, x_2) \in \mathbb{R}^2 / |x_1 - a_1| < \delta$ and $|x_2 - a_2| < \delta\}$

Note: (1) Every spherical neighbourhood of a point in $\mathbb{R}^n$ contains a rectangular neighbourhood of a point and viceversa.

* δ – neighbourhood of a point:

Let $a = (a_1, a_2, \cdots a_n) \in \mathbb{R}^n$ and 'δ' be a +ve real number. The set of points $x = (x_1, x_2 \cdots x_n) \in \mathbb{R}^n$ where $a_r - \delta < x_r < a_r + \delta$

i.e. $|x_r - a_r| < \delta$; $\forall r = 1, 2 \cdots n$ is called a δ-neighbourhood of the point 'a' and is denoted by $N(a, \delta)$.

– If we exclude the point 'a' from $N(a, \delta)$ then it is called a deleted neighbourhood of 'a' and is denoted by $N'(a, \delta)$.

* Limit of a function:–

Let $f: X \to \mathbb{R}$, $X \subset \mathbb{R}^n$. Then f is said to tend to limit $l \in \mathbb{R}$ as 'x' approaches 'a'.

i.e. $\lim_{x \to a} f(x) = l$

i.e. for a given $\epsilon > 0$, $\exists$ a $\delta > 0$.

such that $|f(x) - l| < \epsilon$ whenever $0 < \|x - a\| < \delta$ (or) $0 < |x_r - a_r| < \delta$.

Note:– (1) The limit of a function $f($ as $x \to a$, if it exists at all, is unique

(2) If $x \to a$ where $n \geq 2$ then x approaches 'a' along infinitely many ways unlike the case of $n=1$ when x approaches 'a' along two ways only (i.e. $x \to a-$ & $x \to a+$)

Further for $n \geq 2$, x may approaches along straight lines (or) along differe curves.

In the case of $n=1$, the existence of limit of $f(x)$ as $x \to a$ is independent of the two approaches.

In the case $n \geq 2$ also, the existence of limit is independent of infinitely many approaches. –

(3) For a function of two variables i.e. in the case of $n=2$, two approaches are of special importance. These are

(i) x approaches 'a' first along a line $\parallel^{led}$ to the first axis, then along a line $\parallel^{led}$ to the second axis.

(ii) x approaches 'a' first ~~along a line~~ $\parallel^{le}$ to the second axis and then along a line $\parallel^{led}$ to the first axis.

The limits, if they exist, in these approaches are called repeated limits (or) iterated limits.

(i.e. the function $f(x,y)$ of two variables x and y and (x_0, y_0) is the limiting point of a set of values in two dimensional space.

If $\underset{y\to y_0}{Lt} f(x,y) = g(x)$ then $\underset{x\to x_0}{Lt}\underset{y\to y_0}{Lt} f(x,y)$ is defined by $\underset{x\to x_0}{Lt}\, g(x)$.

If $\underset{x\to x_0}{Lt} f(x,y) = h(y)$ then $\underset{y\to y_0}{Lt}\underset{x\to x_0}{Lt} f(x,y)$ is defined by $\underset{y\to y_0}{Lt}\, h(y)$.

The limits (if exist) $\underset{x\to x_0}{Lt}\underset{y\to y_0}{Lt} f(x,y)$ and $\underset{y\to y_0}{Lt}\underset{x\to x_0}{Lt} f(x,y)$ are the repeated limits.

The limit defined above that is independent of the different approaches is referred to as the double limit (or) Simultaneous limit to distinguish the two approaches.

[i.e. we say that the simultaneous limit exists and is equal to l as $(x,y)\to(x_0,y_0)$.

Symbolically written as

$\underset{(x,y)\to(x_0,y_0)}{Lt} f(x,y) = l$ if for given $\epsilon>0$ (however small) $\exists$ a $\delta>0$ (depending on ϵ) such that $|f(x,y)-l|<\epsilon$ whenever $0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$

$|x-x_0|<\delta,\ |y-y_0|<\delta$.]

Note:

(i) The double (simultaneous) limit may exist but the repeated limits may not exist, but if they exist they must be equal to the double limit.

(ii) The repeated limits may exist but the double limit may not exist.

(iii) If the repeated limits are not equal, the simultaneous limit cannot exist.

*Non-Existence of Simultaneous limit:

For the existence of simultaneous limit, not only must we have same limiting value if the variable point (x,y) approaches the limiting point (x_0,y_0) through any set of values dense at the point, but we must also have the same limiting value as the variable point approaches its limiting position along any curve what so ever.

Thus, if we can find two methods of approach to the limiting point, which give different limiting values then we can conclude that the simultaneous limit does not exist.

Problems:

→ Show that the simultaneous limit

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$$ doesnot exist.

Sol^n: Let $f(x,y) = \frac{xy^3}{x^2+y^6}$; $(x,y)\neq(0,0)$

If we approach the origin along any axis then $\lim_{x\to 0} f(x,0) = 0 = \lim_{y\to 0} f(0,y)$

$f(x,y) \to 0$ as

$(x,y) \to (0,0)$ along coordinate axes.

If we approach $(0,0)$ along a straight line path $y=mx$.

$$f(x,mx) = \frac{xm^3x^3}{x^2+m^6x^3} = \frac{m^3x^2}{1+m^6x}$$

$$\lim_{x\to 0} f(x,mx) = 0$$

$\therefore f(x,y) \to 0$ as $(x,y) \to (0,0)$ along a straightline path.

If we approach $(0,0)$ along the curve $x = my^3$

$$\therefore f(my^3,y) = \frac{my^3y^3}{m^2y^6+y^6}$$

$$= \frac{m}{1+m^2}$$

$$\therefore \lim_{y\to 0} f(my^3,y) = \frac{m}{1+m^2} \neq 0.$$

Since the limit dependence upon the value of m.

$\therefore f(x,y)$ approaches different value along the different curves.

$\therefore$ The limit at the origin doesnot exist.

Note: The existence of the simultaneous limit

$\lim_{(x,y)\to(x_0,y_0)} f(x,y) \Rightarrow$ the single limits

$\lim_{x\to x_0} f(x,y_0)$, $\lim_{y\to y_0} f(x_0,y)$

also exist.

However, it doesnot follow the single limits $\lim_{x\to x_0} f(x,y)$, $\lim_{y\to y_0} f(x,y)$ exist for $y\neq y_0$, $x\neq x_0$ respectively.

→ Show that the simultaneous limit $\lim_{(x,y)\to(0,0)} y\sin(\frac{1}{x})$ exists and equal to but the single limit $\lim_{x\to 0} y_1 \sin\frac{1}{x}$ $(y_1\neq$ doesnot exist.

Sol^n:- Let $\epsilon>0$ be given,

Now we have $\left|y\sin\frac{1}{x} - 0\right| = \left|y\sin\frac{1}{x}\right|$

$$= |y|\left|\sin\frac{1}{x}\right|$$

$$= |y| \quad (\because \left|\sin\frac{1}{x}\right| \leq 1)$$

$$< \epsilon$$

whenever $0 < |y| < \frac{\epsilon}{1}$

$= \delta$ (choosing)

$\therefore \left|y\sin\frac{1}{x} - 0\right| < \epsilon$ whenever $0<|x|<\delta$

$0<|y|<\delta$

$$\therefore \lim_{(x,y)\to(0,0)} y\sin\frac{1}{x} = 0.$$

but for any constant value of $y = y_1 \neq 0$, we get

$$\lim_{x\to 0} y_1 \sin\frac{1}{x} = y_1 \lim_{x\to 0} \sin\frac{1}{x}$$ which doesnot exist.

→ show that $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4}$ doesnot exist.

Sol'n: If we put $x = my^2$ and let $y\to 0$

$$\therefore \lim_{y\to 0} \frac{2my^4}{(m^2+1)y^4} = \frac{2m}{(m^2+1)}$$ doesnot exist

(∵ the limit dependence upon the value of m).

→ show that $\lim_{(x,y)\to(0,0)} xy\frac{x^2-y^2}{x^2+y^2} = 0$.

Sol'n:- put $x = r\cos\theta$; $y = r\sin\theta$

$$\left|xy\frac{x^2-y^2}{x^2+y^2}\right| = \left|r^2\cos\theta\sin\theta(\cos 2\theta)\right|$$

$$= \left|\frac{r^2}{2}\sin 2\theta\cos 2\theta\right|$$

$$= \left|\frac{r^2}{4}2\sin 2\theta\cos 2\theta\right|$$

$$= \left|\frac{r^2}{4}\sin(4\theta)\right|$$

$$\le \frac{r^2}{4} = \frac{x^2+y^2}{4}$$

$$= \frac{x^2}{4} + \frac{y^2}{4}$$

$$< \epsilon$$

if $\frac{x^2}{4} < \frac{\epsilon}{2}$, $\frac{y^2}{4} < \frac{\epsilon}{2}$

i.e. if $|x| < \sqrt{2\epsilon} = \delta$, $|y| < \sqrt{2\epsilon} = \delta$

∴ for $\epsilon > 0$, ∃ a $\delta > 0$.

Such that $\left|xy\frac{x^2-y^2}{x^2+y^2} - 0\right| < \epsilon$

whenever $0 < |x| < \delta$
$0 < |y| < \delta$

$$\therefore \lim_{(x,y)\to(0,0)} xy\frac{x^2-y^2}{x^2+y^2} = 0$$

→ show that repeated limits exists when $(x,y)\to(0,0)$

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} ; & (x,y)\neq(0,0); \\ & (x,y)=(0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$$

Sol'n: $\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0} (0) = 0$

and $\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0}(0) = 0$.

∴ The repeated limit exists and are equal. But the simultaneous limit does not exist by putting $y = mx$.

→ $f(x,y) = \frac{y-x}{y+x}\cdot\frac{1+x}{1+y}$ then

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0}\left[(-1)\left(\frac{1+x}{1}\right)\right] = -1$$

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0}\left(\frac{1}{1+y}\right) = 1$$

∴ The repeated limits exist ~~but~~ are not equal.

∴ The simultaneous limit does not exist.

→ Show that the simultaneous limit exists at the origin

$$f(x,y) = \begin{cases} x\left(\sin\frac{1}{y}\right) + y\sin\left(\frac{1}{x}\right) ; & xy \neq 0 \\ 0 ; & xy = 0 \end{cases}$$

Sol'n: Here $\lim_{y\to 0} f(x,y)$, $\lim_{x\to 0} f(x,y)$ do not exist.

$\therefore \lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$; $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$ do not exist.

Now $|f(x,y)-0| = \left|x\sin\frac{1}{y} + y\sin\frac{1}{x}\right|$

$$\leq |x|+|y| < \epsilon$$

whenever $0<|x|<\epsilon/2$; $0<|y|<\epsilon/2$

Choosing $\delta = \epsilon/2$

$\therefore |f(x,y)-0|<\epsilon$ whenever $0<|x|<\delta$; $0<|y|<\delta$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

→ Show that the repeated limits exist at the origin and are equal but the simultaneous limit does not exist, where $f(x,y) = \begin{cases} 1 & \text{if } xy\neq 0 \\ 0 & \text{if } xy=0 \end{cases}$

Sol^n: $\lim\limits_{y\to 0} f(x,y) = 1 = \lim\limits_{x\to 0} f(x,y)$

$\therefore \lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y) = 1 = \lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$

$\therefore$ Repeated limits exist and are equal.

Let $(x,y)\to(0,0)$ along the coordinate axes.

$$\lim_{x\to 0} f(x,0) = 0 = \lim_{y\to 0} f(0,y)$$

$\therefore f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the coordinate axes.

Let $(x,y)\to(0,0)$ along any other path.

$$\lim_{(x,y)\to(0,0)} f(x,y) = 1$$

$\therefore \lim\limits_{(x,y)\to(0,0)} f(x,y)$ doesnot exist.

→ Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ and $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$ exist but $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$ doesnot exist.

where $f(x,y) = \begin{cases} y + x\sin(1/y) & \text{if } y\neq 0 \\ 0 & \text{if } y=0. \end{cases}$

Sol^n:— Here $\lim\limits_{y\to 0} f(x,y)$ does not exist.

$\therefore \lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$ does not exist.

Now $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y) = \lim\limits_{y\to 0} y$

$$= 0$$

$\therefore \lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$ exists and is equal to 0

and now $|f(x,y)-0| = \left|y + x\sin\frac{1}{x}\right|$

$$\leq |y|+|x| \quad \left(\because \left|\sin\frac{1}{y}\right|\leq 1\right)$$

$< \epsilon$ whenever

$0<|x|<\epsilon/2$; $0<|y|<\epsilon/2$

Choosing $\epsilon/2 = \delta$

$\therefore |f(x,y)-0|<\epsilon$ whenever $0<|x|<\delta$; $0<|y|<\delta$.

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

* Algebra of Limits:—

If f, g are two functions defined on some neighbourhood of a point (a,b) such that $\lim\limits_{(x,y)\to(a,b)} f(x,y) = l$, $\lim\limits_{(x,y)\to(a,b)} g(x,y) = m$ then (i) $\lim (f\pm g) = \lim f \pm \lim g = l \pm g$

(2). $\lim (f.g) = \lim f . \lim g$
$= l.m$

(3) $\lim (f/g) = \frac{\lim f}{\lim g} = \frac{l}{m}$ provided $m \neq 0$

when $(x,y) \to (a,b)$.

Problems

→ $\lim_{(x,y)\to(1,2)} (x^2+2y) = \lim_{(x,y)\to(1,2)} x^2 + \lim_{(x,y)\to(1,2)} 2y$
$= 1+2(2)$
$= 1+4$
$= 5$

→ $\lim_{(x,y)\to(0,0)} \frac{x\sin(x^2+y^2)}{x^2+y^2}$

$= \lim_{(x,y)\to(0,0)} x \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{(x^2+y^2)}$

$= 0.1 = 0$

→ $\lim_{(x,y)\to(2,1)} \frac{\sin^{-1}(xy-2)}{\tan^{-1}(3xy-6)}$

$= \lim_{t\to0} \frac{\sin^{-1}(t)}{\tan^{-1}(3t)}$ [Put $xy-2 = t$ and $(x,y) \to (2,1)$ $\Rightarrow t \to 0$]

$= \lim_{t\to0} \frac{\frac{1}{\sqrt{1-t^2}}}{\frac{3}{1+9t^2}}$

$= \lim_{t\to0} \frac{1}{3} \cdot \frac{1+9t^2}{\sqrt{1-t^2}}$

$= \frac{1}{3}$

*Continuity:—
→ A function $f(x,y)$ is said to be continuous at a point (a,b) if

$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

i.e. if for given $\epsilon>0$, $\exists$ a $\delta>0$ such that $|f(x,y)-f(a,b)| < \epsilon$ whenever $|x-a|<\delta$, $|y-b|<\delta$

→ If f is not continuous at (a,b), $(a,b) \in D \subset \mathbb{R}^2$ then f is said to be discontinuous at (a,b).

→ f is said to be continuous on the domain D, if f is continuous at each point of D.

Note:-
Let $D \subset \mathbb{R}^2$ and $f: D \to \mathbb{R}$ be continuous function at $(a,b) \in D$.

Let $f_1(x) = f(x,b)$ then f is a function of single variable 'x'.

Since $f(x,y)$ is continuous at (a,b).

$\Rightarrow \lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

i.e. given $\epsilon>0$, $\exists$ a $\delta>0$ such that $|f(x,y)-f(a,b)| < \epsilon$ whenever $|x-a|<\delta$, $|y-b|<\delta$; $(x,y) \in D$.

$\Rightarrow |f(x,b)-f(a,b)| < \epsilon$ whenever $|x-a|<\delta$; $(x,b) \in D$.

$\Rightarrow |f_1(x)-f_1(a)| < \epsilon$ whenever $|x-a|<\delta$; $(x,b) \in D$.

$\Rightarrow f_1$ is continuous at 'a'.

Similarly, we show that $f_2(y) = f(a,y)$ is continuous at 'b'.

$\therefore$ If $f(x,y)$ is continuous at (a,b) then

(i) $f(x,b)$ is continuous at $x=a$ and

(ii) $f(a,y)$ is continuous at $y=b$.

But the converse of above is not true

i.e. if $f(x,b)$ is continuous at $x=a$ and $f(a,y)$ is continuous at $y=b$. then $f(x,y)$ need not be continuous at (a,b).

Problem :-

(1) Examine the continuity at $(1,2)$ of the function

$$f(x,y) = \begin{cases} x^2+4y & \text{when } (x,y)\neq(1,2) \\ 0 & \text{when } (x,y)=(1,2) \end{cases}$$

Sol^n: $\lim_{(x,y)\to(1,2)} f(x,y) = \lim_{(x,y)\to(1,2)} (x^2+4y)$

$= \lim_{(x,y)\to(1,2)} x^2 + \lim_{(x,y)\to(1,2)} 4y$

$= 1^2+4(2)$

$= 1+8 = 9$ and $f(1,2)=0$

$f(1,2)=c$

$\therefore \lim_{(x,y)\to(1,2)} f(x,y) \neq f(1,2)$.

$\therefore f(x,y)$ is not continuous at $(1,2)$

Show that the function $f: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{(y^2-x^2)yx}{x^2+y^2} & \text{for } (x,y)\neq(0,0) \\ 0 & \text{for } (x,y)=(0,0) \end{cases}$$

is continuous at $(0,0)$.

Sol^n: Let $\epsilon>0$ given, Now we have

$$|f(x,y)-f(0,0)| = \left|\frac{(y^2-x^2)yx}{x^2+y^2} - 0\right|$$

$$= \left|\frac{(y^2-x^2)}{x^2+y^2}\, yx\right|$$

$$= \left|\frac{y^2-x^2}{x^2+y^2}\right| |xy|$$

$\leq |xy|$ $\left\{\because \left|\frac{y^2-x^2}{x^2+y^2}\right| \leq 1 \text{ for } f(x,y)\neq(\right.$

$= |x|\,|y|$

$< \epsilon$ whenever $|x|<\sqrt{\epsilon}$ & $|y|<\sqrt{\epsilon}$

choosing $\sqrt{\epsilon} = \delta$

$\therefore |f(x,y)-f(0,0)|<\epsilon$ whenever $|x|<\delta$, $|y|<\delta$.

$\therefore f(x,y)$ is continuous at $(0,0)$

(2) Discuss the continuous of the function

$$f(x,y) = \begin{cases} \dfrac{2xy^2}{x^3+3y^3} ; & (x,y)\neq(0,0) \\ 0 ; & (x,y)=(0,0) \end{cases}$$

Sol^n: $f(0,0)=0$

Let $(x,y)\to(0,0)$ along the coordinate axes then $\lim_{x\to0} f(x,0)=0=\lim_{y\to0} f(0,y)$

$\therefore f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the coordinate axes.

Let $(x,y)\to(0,0)$ along straight line $y=x$ then $\lim_{x\to0} f(x,x) = \lim_{x\to0} \frac{2x\cdot x^2}{x^3+3x^3}$

$= \lim_{x\to0} \frac{2}{4} = \frac{1}{2}$.

$\therefore f(x,y)\to\frac{1}{2}$ as $(x,y)\to(0,0)$ along the straight line path.

Since the two methods of approach to the limiting points give different limiting values.

$\therefore$ the simultaneous limits donot exist.

i.e, $\underset{(x,y)\to(0,0)}{Lt} f(x,y)$ does not exist

$\therefore f(x,y)$ is not continuous at $(0,0)$.

$\rightarrow$ Show that function

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} ; & (x,y) \neq (0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$$ is

continuous at $(0,0)$.

Sol'n: Let $\epsilon > 0$ be given

Now we have $|f(x,y) - f(0,0)| = \left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right|$

$$= \left|\frac{xy}{\sqrt{x^2+y^2}}\right|$$

$$= |xy| \cdot \frac{1}{\sqrt{x^2+y^2}}$$

$$= \left|\frac{xy}{x^2+y^2}\right| \sqrt{x^2+y^2}$$

$$\leq \frac{1}{2}\sqrt{x^2+y^2} \quad [\because 2|xy| \leq x^2+y^2$$

$$\Rightarrow \left|\frac{xy}{x^2+y^2}\right| \leq \frac{1}{2}$$

if $(x,y) \neq (0,0)$]

$$< \sqrt{x^2+y^2}$$

$$< \epsilon$$

whenever $x^2+y^2 < \epsilon^2 = \delta$ (choosing)

$\therefore |f(x,y) - f(0,0)| < \epsilon$ whenever $x^2+y^2 < \epsilon^2$

$\therefore f(x,y)$ is continuous at $(0,0)$

(or)

Let $x = r\cos\theta$; $y = r\sin\theta$.

$$|f(x,y) - f(0,0)| = \frac{xy}{\sqrt{x^2+y^2}}$$

$$= \frac{|r^2 \sin\theta \cos\theta|}{r}$$

$$= r|\sin\theta||\cos\theta|$$

$$\leq r \quad (\because |\sin\theta| \leq 1 \ \& \ |\cos\theta| \leq 1)$$

$$= \sqrt{x^2+y^2}$$

$$< \epsilon \text{ whenever } x^2+y^2 < \epsilon^2 = \delta.$$

$\therefore |f(x,y) - f(0,0)| < \epsilon$ whenever $x^2+y^2 < \delta$

$\therefore f(x,y)$ is continuous at $(0,0)$.

H.W. Show that the following functions are discontinuous at $(0,0)$

(i) $f(x,y) = \begin{cases} \frac{1}{x^2+y^2} ; & (x,y) \neq (0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$

(ii) $f(x,y) = \begin{cases} \frac{x^4-y^4}{x^4+y^4} ; & (x,y) \neq 0 \\ 0 ; & (x,y) = (0,0) \end{cases}$

(iii) $f(x,y) = \begin{cases} \frac{x^2y^2}{x^4+y^4} ; & (x,y) \neq (0,0) \\ 0 ; & (x,y) \neq (0,0) \end{cases}$

$\rightarrow$ Show that the following functions are continuous at the origin.

(i) $f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} ; & (x,y) \neq (0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$

(ii) $f(x,y) = \begin{cases} \frac{x^3y^3}{x^2+y^2} ; & (x,y) \neq (0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$

(iii) $f(x,y) = \begin{cases} \frac{x^3+y^3}{x^2+y^2} ; & (x,y) \neq (0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$

$\rightarrow$ Discuss the following function for continuity at $(0,0)$

$$f(x,y) = \begin{cases} \frac{x^2y}{x^3+y^3} ; & x^3+y^3 \neq 0 \text{ i.e. } (x,y) \neq (0,0) \end{cases}$$

Partial Derivatives

The ordinary derivative of a function of several variables with respect to one of the independent variables, keeping all other independent variables constant is called the partial derivative of the function w.r.t the variable.

→ Partial derivative of $f(x, y)$ w.r.t x denoted by $\frac{\partial f}{\partial x}$ or f_x or $f_x(x, y)$, while those w.r.t y denoted by $\frac{\partial f}{\partial y}$ or f_y or $f_y(x, y)$.

$$\therefore \frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

and
$$\frac{\partial f}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y}$$
when these limits exist.

→ The partial derivatives at a particular point (a, b) are often denoted by $\left(\frac{\partial f}{\partial x}\right)_{(a,b)}$, $\frac{\partial f(a,b)}{\partial x}$ or $f_x(a, b)$ and

$\left(\frac{\partial f}{\partial y}\right)_{(a,b)}$, $\frac{\partial f(a,b)}{\partial y}$, $f_y(a, b)$

$$\therefore f_x(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

and
$$f_y(a, b) = \lim_{k \to 0} \frac{f(a, b+k) - f(a, b)}{k}$$

Note: we have, by definition

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}$$
$$= \lim_{x \to a} \frac{f(x, b) - f(a, b)}{x - a}$$

and

$$f_y(a,b) = \lim_{k\to 0} \frac{f(a,b+k)-f(a,b)}{k}$$

$$= \lim_{y\to b} \frac{f(a,y)-f(a,b)}{y-b}$$

problems:

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function given by

$f(x,y) = x^2+xy+y^3$.

Find $f_x(x,y)$ and $f_y(x,y)$.

Soln: By definition,

$$f_x(x,y) = \frac{f(x+h,y)-f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2+(x+h)y+y^3-x^2-xy-y^3}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2+\cancel{xy}+hy+\cancel{y^3}-\cancel{x^2}-\cancel{xy}-\cancel{y^3}}{h}$$

$$= \lim_{h\to 0} \frac{2xh+h^2+hy}{h}$$

$$= \lim_{h\to 0} 2x+h+y$$

$$= 2x+y$$

Similarly

$$f_y(x,y) = \lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k}$$

$$= \lim_{k\to 0} \frac{\cancel{x^2}+\cancel{x}(y+k)+(y+k)^3-\cancel{x^2}-\cancel{xy}-y^3}{k}$$

$$= \lim_{k\to 0} \frac{xk+3y^2k+3k^2y+k^3}{k}$$

$$= \lim_{k\to 0} x+3y^2+3ky+k^2$$

$$= x+3y^2$$

$f: \mathbb{R}^3 \to \mathbb{R}$ be a function defined by

$f(x,y,z) = xy+yz+zx$.

the partial derivatives f_x, f_y, f_z at (a,b,c).

Given, $f(x,y,z) = xy+yz+zx$.

definition

$$f_x(a,b,c) = \underset{h\to 0}{Lt} \frac{f(a+h,b,c) - f(a,b,c)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{(a+h)b + bc + c(a+h) - ab - bc - ca}{h}$$

$$= \underset{h\to 0}{Lt} \frac{hb + ch}{h}$$

$$= b+c$$

$$f_y(a,b,c) = \underset{k\to 0}{Lt} \frac{f(a,b+k,c) - f(a,b,c)}{k}$$

$$= \underset{k\to 0}{Lt} \frac{a(b+k) + (b+k)c + ca - ab - bc - c}{k}$$

$$= a+c$$

$$f_z(a,b,c) = \underset{l\to 0}{Lt} \frac{f(a,b,c+l) - f(a,b,c)}{l}$$

$$= \underset{l\to 0}{Lt} \frac{ab + b(c+l) + (c+l)a - ab - bc - c}{l}$$

$$= b+a$$

→ Let $f: \mathbb{R}^n \to \mathbb{R}$ be a function defined by

$f(x_1, x_2, \dots, x_n) = x_1^2 + x_2^2 + \dots + x_n^2$,

find the f_{x_i} at the point $(a_1, a_2, \dots, a_n)$.

$(a_1, a_2, \dots, a_n)$.

Sol: - To find the partial derivative of f w.r.t x_i at the point $(a_1, a_2, \dots, a_n)$;

we write

$$f_{x_i}(a_1, a_2, \cdots a_n) = \lim_{h\to 0} \frac{f(a_1, a_2, \cdots\cdots a_i+h, a_{i+1} \cdots a_n) - f(a_1, a_2, \cdots a_n)}{h}$$

$$= \lim_{h\to 0} \frac{a_1^2 + a_2^2 + \cdots + a_{i-1}^2 + (a_i+h)^2 + a_{i+1}^2 + \cdots + a_n^2 - (a_1^2 + a_2^2 + \cdots a_n^2)}{h}$$

$$= \lim_{h\to 0} \frac{2a_ih + h^2}{h} = 2a_i$$

Note: $f_x(x,y)$ is nothing but the derivative of $f(x,y)$ considered as a function of a single variable x, treating y as a constant.

Similarly $f_y(x,y)$ is nothing but the derivative of $f(x,y)$ considering it as a function of the single variable y, and treating x as a constant.

In general, $f_{x_i}(x_1, x_2, \cdots x_n)$ is the derivative of $f(x_1, x_2, \cdots x_n)$ w.r.t x_i treating all the other variables as constants.

→ Let us find the partial derivatives of the following functions.

(i) $f = x^3 - 4x^2y^2 + 8y^2$

(ii) $f = x\sin y + y\cos x$

(iii) $f = xe^y + ye^x$.

Sol^n: In all the three cases, the functions involved are either polynomials or

[illegible] trigonometric or exponential functions. [illegible] ensures that the partial derivatives exist. (∵ the polynomial, trigonometric and exponential functions of single variable are differentiable).

By direct differentiation, we get.

(i) $\frac{\partial f}{\partial x} = 3x^2 - 8xy^2$

and $\frac{\partial f}{\partial y} = -8x^2y + 16y^3$

(ii) $\frac{\partial f}{\partial x} = \sin y - y\sin x$

and $\frac{\partial f}{\partial y} = x\cos y + \cos x$

(iii) $\frac{\partial f}{\partial x} = e^y + ye^x$

and $\frac{\partial f}{\partial y} = xe^y + e^x$

→ If $f(x,y) = 2x^2 - xy + 2y^2$. Then find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(1,2)$.

Sol^n: $\frac{\partial f}{\partial x} = 4x - y = 2$ at $(1,2)$.

$\frac{\partial f}{\partial y} = -x + 4y = 7$ at $(1,2)$.

Note: The calculation of partial derivatives is not always as simple as in these examples. In some exceptional cases, we have to use the limiting process.

Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} \frac{xy}{x^4+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Find the two partial derivatives at the points $(0,0)$, $(a,0)$, $(0,b)$ and (a,b) where $a \neq 0$, $b \neq 0$.

Sol^n: By definition

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{0-0}{k} = 0$$

$$f_x(a,0) = \lim_{h\to 0} \frac{f(a+h,0) - f(a,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_y(a,0) = \lim_{k\to 0} \frac{f(a,0+k) - f(a,0)}{k} = \lim_{k\to 0} \frac{f(a,k) - f(a,0)}{k}$$

$$= \lim_{k\to 0} \frac{\frac{ak}{a^4+k^4} - 0}{k} = \lim_{k\to 0} \frac{a}{a^4+k^4}$$

$$= \frac{a}{a^4} = \frac{1}{a^3}$$

$$f_x(0,b) = \lim_{h\to 0} \frac{f(0+h,b) - f(0,b)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{bh}{b^4+h^4} - 0}{h} = \frac{1}{b^3}$$

$$f_y(0,b) = \lim_{k\to 0} \frac{f(0,b+k) - f(0,b)}{k}$$

$$= \lim_{k\to 0} \frac{0-0}{k} = 0.$$

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(a+h)b}{(a+h)^4+b^4} - \frac{ab}{a^4+b^4}}{h}$$

$$= \lim_{h\to 0} \frac{(ab+hb)(a^4+b^4) - (ab)\left[(a+h)^4+b^4\right]}{h(a^4+b^4)\left[(a+h)^4+b^4\right]}$$

$$= \lim_{h\to 0} \frac{b^5-3a^4b}{(a^4+b^4)^2}$$

$$f_y(a,b) = \lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k}$$

$$= \lim_{k\to 0} \frac{\frac{a(b+k)}{a^4+(b+k)^4} - \frac{ab}{a^4+b^4}}{k}$$

$$= \frac{a^5-3ab^4}{(a^4+b^4)^2}$$

<u>Note</u>: In the above problem, by direct differentiation, we could have obtained $f_x(a,b)$ and $f_y(a,b)$ $(a,b) \neq (0,0)$ correctly, but not $f_x(0,0)$ or $f_y(0,0)$.

because f is defined as a quotient of two polynomial functions for all $(x,y) \neq (0,0)$; we can use direct differentiation to calculate the partial derivatives at these points. But to calculate $f_x(0,0)$ or $f_y(0,0)$ we need to use $f(0,0)$, which is not defined.

→ If $f: \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} \frac{x}{y} + \frac{y}{x}, & y \neq 0, x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

then show that $f_x(0,1)$ and $f_y(1,0)$ do not exist.

Soln: By definition

$$f_x(0,1) = \lim_{h \to 0} \frac{f(0+h, 1) - f(0,1)}{h}$$

$$= \lim_{h \to 0} \frac{f(h,1) - f(0,1)}{h}$$

$$= \lim_{h \to 0} \frac{h + \frac{1}{h} - 0}{h}$$

$$= \lim_{h \to 0} \frac{h\left[1 + \frac{1}{h^2}\right]}{h}$$

$$= \lim_{h \to 0} \left(1 + \frac{1}{h^2}\right) = \infty$$

and $$f_y(1,0) = \lim_{k \to 0} \frac{f(1, 0+k) - f(1,0)}{k}$$

$$= \lim_{k \to 0} \frac{\frac{1}{k} + k - 0}{k}$$

$$= \lim_{k \to 0} \frac{1}{k^2} + 1$$

$$= \infty$$

$\therefore$ $f_x(0,1)$ and $f_y(1,0)$ do not exist.

Note: The existence of partial derivatives at a point need not imply continuity at that point.

For example:

2004. If $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

Show that both the partial derivatives exist at $(0,0)$ but the function is not continuous at $(0,0)$.

Soln: $f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h}$

$= \lim_{h \to 0} \frac{0-0}{h}$

$= 0$

$f_y(0,0) = \lim_{k \to 0} \frac{f(0,0+k) - f(0,0)}{k}$

$= \lim_{k \to 0} \frac{0-0}{k}$

$= 0$

$\therefore f$ possesses both the partial derivatives at $(0,0)$.

Now let $(x,y) \to (0,0)$ along the straight line $y = mx$.

$\lim_{x \to 0} f(x, mx) = \lim_{x \to 0} \frac{mx^2}{x^2(1+m^2)}$

$= \frac{m}{1+m^2}$

which depends upon m.

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ doesnot exist.

$\therefore f(x,y)$ is not continuous at $(0,0)$

2011 Find $f_x(0,0)$ and $f_y(0,0)$ & $f_y(0,0)$

if $f(x,y) = \begin{cases} \dfrac{x^2 - xy}{x+y} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

(Q) If $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y} & ; x \neq y \\ 0 & ; x = y. \end{cases}$

then show that f is discontinuous at the origin but the partial derivatives exist at the origin.

Sol: Let $(x,y) \to (0,0)$ along the curve $y = x - mx^3$.

2005 Show that the given below is not continuous at $(0,0)$

$f(x,y) = \begin{cases} 0 & \text{if } xy = 0 \\ 1 & \text{if } xy \neq 0 \end{cases}$

Sol: Let $(x,y) \to (0,0)$ along the co-ordinate axes.

$\lim\limits_{x\to 0} f(x,0) = \lim\limits_{x\to 0} 0 = 0$

and $\lim\limits_{y\to 0} f(0,y) = \lim\limits_{y\to 0} 0 = 0$

$\therefore \lim\limits_{x\to 0} f(x,0) = 0 = \lim\limits_{y\to 0} f(0,y)$

$\therefore f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the co-ordinate axes.

Let $(x,y) \to (0,0)$ along any other path

$\lim\limits_{(x,y)\to(0,0)} f(x,y) = 1$

For example:

(1) Let $f: \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x_1, x_2, x_3) = |x_1| + |x_2| + |x_3|.$$

Show that f is continuous at $(0,0,0)$ but does not possess any of the three first order partial derivatives at $(0,0,0)$.

Sol:

Now at the point $(0,0,0)$, we have

$$\frac{f(0+h,0,0) - f(0,0,0)}{h}$$

$$= \frac{|h|}{h} = f_1(h) \text{ (say)}$$

$$\therefore \lim_{h \to 0^+} f_1(h) = \lim_{h \to 0^+} \frac{|h|}{h}$$

$$= 1.$$

But $\lim_{h \to 0^-} f_1(h) = \lim_{h \to 0^-} \frac{|h|}{h} = -1$.

Hence $\lim_{h \to 0} f_1(h)$ does not exist.

Similarly, $\lim_{h \to 0} f_2(h)$ and $\lim_{h \to 0} f_3(h)$ also do not exist.

$\therefore f$ does not possess any of the first order partial derivatives at the point $(0,0,0)$.

But this function is continuous at $(0,0,0)$.

Since the two methods of approach to the limiting point give different limiting values.

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ doesnot exist.

$\therefore f(x,y)$ is not continuous at $(0,0)$.

2006 P-II → Show that the function given by

$$f(x,y) = \begin{cases} \dfrac{x^3+2y^3}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0). \end{cases}$$

(i) is continuous at $(0,0)$

(ii) possesses partial derivatives $f_x(0,0)$ & $f_y(0,0)$

2007 P-II → Show that the function given by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+2y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0). \end{cases}$$

is not continuous at $(0,0)$ but partial derivatives f_x & f_y exist at $(0,0)$.

→ Examine the continuity of the function $f(x,y) = \sqrt{|xy|}$ at the origin.

Note: We know that a real valued continuous function of a real variable need not be differentiable. The same is true for functions of several variables.

i.e, function of several variables which is continuous at a point need not have any of the partial derivatives at the point.

We have seen that the existence of partial derivatives does not imply continuity. However, if partial derivatives satisfy some more conditions, then we can ensure continuity. In order to prove this theorem we need a simple result which follows easily from Lagrange's mean value theorem.

Mean value theorem:

If f_x exists throughout a nbd of a point (a,b) and $f_y(a,b)$ exists then for any point $(a+h, b+k)$ of this nbd,

$$f(a+h, b+k) - f(a,b) = h f_x(a+\theta h, b+k) + k(f_y(a,b) + \eta)$$

where $0<\theta<1$ and η is a function of k, which tends to '0' as $k \to 0$

Note: we can see that this is an extension of Lagrange's mean value theorem to functions from $\mathbb{R}^2 \to \mathbb{R}$.

Interchanging x and y in the above theorem, the theorem can be written as:

"If f_y exists throughout a nbd of a point (a,b) and $f_x(a,b)$ exists then for any point $(a+h, b+k)$ of this nbd,

$$f(a+h, b+k) - f(a,b) = k f_y(a+h, b+\theta' k) + h(f_x(a,b) + \eta')$$

where $0<\theta'<1$ and η' is a function of h and tends zero as $h \to 0$."

→ **A sufficient condition for continuity:**

A sufficient condition that a function

f be continuous at (a, b) is that one of the partial derivatives exists and is bounded in a nbd of (a, b) and that the other exists at (a, b).

proof:

Let f_x exist ~~and~~ bounded in a nbd of (a, b) and let $f_y(a,b)$ exist.

then for any point $(a+h, b+k)$ of this nbd we have

$$f(a+h, b+k) - f(a,b) = h\, f_x(a+\theta h, b+k) + k[f_y(a,b) + \eta] \quad \text{—(1)}$$

where $0<\theta<1$ and $\eta \to 0$ as $k \to 0$.

proceeding to limits as $(h,k) \to (0,0)$,

Since f_x bounded in a nbd of N,

it follows that

$$\lim_{(h,k)\to(0,0)} h f_x(a+\theta h, b+k) = 0$$

Consequently, from (1), we get

$$\lim_{(h,k)\to(0,0)} f(a+h, b+k) = f(a,b)$$

$\Rightarrow$ f is continuous at (a, b).

Note: A sufficient condition that a function be continuous in a closed region is that both the partial derivatives exist and are bounded throughout the region.

[illegible]f of the mean value theorem

$f(a+h, b+k) - f(a,b) = f(a+h, b+k)$
$- f(a, b+k) + f(a, b+k)$
$- f(a,b)$. ——— (1)

Since f_x exists throughout a nbd of a point (a,b),
therefore by Lagrange's mean value theorem,

$$f_x(a+\theta h, b+k) = \frac{f(a+h, b+k) - f(a, b+k)}{h}, \quad 0<\theta<1$$

$$\Rightarrow f(a+h, b+k) - f(a, b+k) = h\, f_x(a+\theta h, b+k), \quad 0<\theta<1. \quad \text{——— (2)}$$

Also $f_y(a,b)$ exists, so that

$$\lim_{k\to 0} \frac{f(a, b+k) - f(a,b)}{k} = f_y(a,b).$$

$$\Rightarrow \lim_{k\to 0}\left[f(a, b+k) - f(a,b)\right] = \lim_{k\to 0} k\left[f_y(a,b) + \eta(k)\right]$$

where $\eta(k)$ is a function of k

$$\therefore f(a, b+k) - f(a,b) = k f_y(a,b) + k\eta \quad \text{——— (3)}$$

where η is a function of k and tends to zero as $k \to 0$.

$\therefore$ From (1), (2) and (3), we have

$$f(a+h, b+k) - f(a,b) = h\, f_x(a+\theta h, b+k) + k\left[f_y(a,b) + \eta\right].$$

which is the required result.

Differentiability:

Let f be a real valued function defined in a nbd N of a point (a,b):

we say that the function f is differentiable at (a,b), if

$$f(a+h, b+k) - f(a,b) = Ah + Bk + h\phi(h,k) + k\psi(h,k).$$

where

– h and k are real numbers such that $(a+h, b+k) \in N$

– A and B are constants independent of h and k but dependent on the function f and the point (a,b)

– ϕ and ψ are two functions tending to zero as $(h,k) \to (0,0)$.

(or)

Let f be a real valued function defined in a nbd N of a point (a,b). Then the function f is said to be differentiable at the point (a,b), if there exist two constants A and B (depending on f and the point (a,b) only) such that

$$f(a+h, b+k) - f(a,b) = Ah + Bk + \sqrt{h^2+k^2}\,\phi(h,k);$$

where $\phi(h,k)$ is real valued function such that $\phi(h,k) \to 0$ as $(h,k) \to (0,0)$.

→ Let $f(x,y) = x^2+y^2$. S.T. f is differentiable at any point (a,b).

Sol: For any two real numbers h and k, we have

$$f(a+h, b+k) - f(a,b) = (a+h)^2 + (b+k)^2 - (a^2+b^2)$$
$$= 2ah + h^2 + 2bk + k^2 \ldots$$

$= 2ah + 2bk + hh + kk.$

If we let $A = 2a$, $B = 2b$, $\phi(h,k) = h$ and $\psi(h,k) = k$.

then
$f(a+h, b+k) - f(a,b) = Ah + Bk + h\phi(h,k) + k\psi(h,k)$
where A and B are constants independent of h & k
$\phi(h,k) \to 0$ and $\psi(h,k) \to 0$ as $(h,k) \to (0,0)$.
$\therefore$ f is differentiable at the point (a,b).

$\to$ Let $f(x,y) = \frac{x}{y}$. Then show that f is differentiable at all points (a,b) in the domain of definition of the function.

Sol: Given $f(x,y) = \frac{x}{y}$.

Since f is not defined for $y = 0$

we take $b \neq 0$.

Let h and k be two real numbers such that $(a+h, b+k)$ is a point in a nbd N of (a,b) which is contained in the domain of f.

Then $b+k \neq 0$.

and $f(a+h, b+k) - f(a,b) = \frac{a+h}{b+k} - \frac{a}{b}$

$= \frac{a}{b+k} - \frac{a}{b} + \frac{h}{b+k}$

$= \frac{-ak}{b(b+k)} + \frac{h}{b+k}$

$= -\frac{ak}{b^2}\left[1 - \frac{k}{b+k}\right] + \frac{h}{b}\left[1 - \frac{h}{b+k}\right]$

$= \frac{1}{b}h - \frac{a}{b^2}k + h\left[\frac{-k}{b(b+k)}\right] + k\left(\frac{ak}{b^2(a+k)}\right)$

Then Set $A = \frac{1}{b}$, $B = \frac{-a}{b^2}$, $\phi(h,k) = \frac{-k}{b(b+k)}$ and $\psi(h,k) = \frac{ak}{b^2(b+k)}$

$\ldots b+k) - f(a,b) = Ah + Bk + h\,\phi(h,k) + k\,\psi(h,k)$

where A and B are constants independent of h and k, $\phi(h,k) \to 0$ and $\psi(h,k) \to 0$ as $(h,k) \to (0,0)$.

Hence f is differentiable at (a,b).

→ Prove that the function given by $f(x,y) = |x| + |y|$ is not differentiable at $(0,0)$.

Sol: Suppose, if possible, that f is differentiable at $(0,0)$.

Then $f(0+h, 0+k) - f(0,0) = Ah + Bk + h\,\phi(h,k) + k\,\psi(h,k)$

where A and B are constants.

$\phi(h,k)$, and $\psi(h,k) \to 0$ as $(h,k) \to (0,0)$.

$\therefore |h| + |k| = Ah + Bk + h\,\phi(h,k) + k\,\psi(h,k)$ — (1)

Let $h = 0$ and $k > 0$, then from (1),

$k = Bk + k\,\psi(0,k) \Rightarrow 1 = B + \psi(0,k)$

Taking limits on both sides as $(h,k) \to (0,0)$

we get $B = 1$, because $\psi(0,k) \to 0$

($|x| = x$ if ...)

Now let $h = 0$ and $k < 0$, then

$-k = Bk + k\,\psi(0,k)$

$-1 = B + \psi(0,k)$

Taking limits on both sides as $(h,k) \to (0,0)$

we get $B = -1$; because $\psi(0,k) \to 0$

$\therefore$ The assumption that the given function is differentiable at $(0,0)$ leads us to the contradiction $B = 1 = -1$.

$\therefore |x| + |y|$ is not differentiable at $(0,0)$.

Theorem Let f be real valued function defined in a nbd N of a point (a,b). If f is differentiable at (a,b), then f is continuous at (a,b).

The above theorem shows that continuity in the two variables is a necessary condition for differentiability. However, it is not a sufficient condition.

For example.

Show that the function defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

is continuous at the origin but is not differentiable.

Sol: we verified it is continuous in the pg. no. 30 back page.

Now we prove that $f(x,y)$ is not differentiable at the origin.

we have

$$f(0+h, 0+k) = \frac{hk}{\sqrt{h^2+k^2}}$$

$$\therefore f(0+h, 0+k) - f(0,0) = 0.h + 0.k + \sqrt{h^2+k^2}\cdot\frac{hk}{h^2+k^2}$$

So that $A = 0$, $B = 0$ and $\phi(h,k) = \dfrac{hk}{h^2+k^2}$

i.e. A and B are independent of h, k.

If we put $k = mh$, then we have

$$\lim_{(h,k)\to(0,0)} \phi(h,k) = \lim_{(h,k)\to(0,0)} \frac{hk}{h^2+k^2}$$

$$= \lim_{h\to 0} \frac{mh^2}{h^2+m^2h^2}$$

$$= \lim_{h\to 0} \frac{mh^2}{h^2(1+m^2)} = \frac{m}{1+m^2}$$

$\therefore$ This limit does not exist since it depends

upon m

$\therefore Lt = \phi(h, k) \neq 0$ as $(h,k) \to (0,0)$.

It follows that the given function is not differentiable at (0,0).

Note: to show that the function is not differentiable, it enough to show that f is not continuous.

For example, The function f

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & ; x^2+y^2 \neq 0 \\ 0 & \text{if } x=y=0 \end{cases}$$

is not differentiable at the origin because it is discontinuous there.

→ Show that the function f, where

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \text{if } x^2+y^2 \neq 0 \\ 0 & \text{if } x=y=0 \end{cases}$$

is not differentiable.

→ show that the following functions are not differentiable at (0,0) by showing that they are discontinuous at (0,0)

(i) $f(x,y) = \begin{cases} \frac{x^2+y^2}{x-y} & x \neq y \\ 1 & x = y \end{cases}$ (2) $f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 2 & (x,y) = (0,0) \end{cases}$

(3) $f(x,y) = \begin{cases} \frac{x^5}{x^4+y^4} & (x,y) \neq (0,0) \\ 3 & (x,y) = (0,0) \end{cases}$

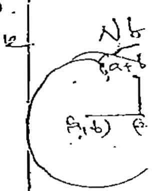
[Hint: $x^4+y^4 \geq x^4 \Rightarrow |f(x,y)| = \left|\frac{x^5}{x^4+y^4}\right| \leq \left|\frac{x^5}{x^4}\right| = |x|$]

→ If f is differentiable at (a,b) then f possesses both the partial derivatives at (a,b).

Sol Since f is differentiable at (a,b)

$\therefore f(a+h, b+k) - f(a,b) = Ah + Bk + h\phi(h,k) + k\psi(h,k)$ — (1)

where A and B are constants depend on f & the point (a,b) and independent of h & k



$\phi(h,k) \to 0$, $\psi(h,k) \to 0$ as $(h,k) \to (0,0)$.

If $(a+h, b+k)$ belonging to the nbd of (a,b)

then $(a+h, b)$ and $(a, b+k)$ also belong to nbd of (a,b)

$\therefore$ If we take $k=0$ in (1) then, we have

$f(a+h, b+0) - f(a,b) = Ah + h\phi(h,0)$.

$\Rightarrow \dfrac{f(a+h,b) - f(a,b)}{h} = A + \phi(h,0)$

$\Rightarrow \lim_{h\to 0} \dfrac{f(a+h,b) - f(a,b)}{h} = A$ $\left(\because \lim_{(h,k)\to(0,0)} \phi(h,0) = 0\right)$

$\Rightarrow \boxed{f_x(a,b) = A} \Rightarrow A = \left(\dfrac{\partial f}{\partial x}\right)_{(a,b)}$.

Similarly we can prove that $B = \left(\dfrac{\partial f}{\partial y}\right)_{(a,b)} = f_y(a,b)$

From this we see that for small values of h and k we can approximate $f(a+h, b+k) - f(a,b)$ by the expression $h f_x(a,b) + k f_y(a,b)$

i.e $f(a+h, b+k) - f(a,b) \approx h f_x(a,b) + k f_y(a,b)$.

Defⁿ: Let $f(x,y)$ be a real-valued function defined in a nbd of the point (a,b). then

If $f(x,y)$ is differentiable at (a,b) then the linear function $T: \mathbb{R}^2 \to \mathbb{R}$ defined by $T(h,k) = h f_x(a,b) + k f_y(a,b)$, is called the differential of f at (a,b) and is denoted by $df(a,b)$.

Note!

The converse of the above need not be true. i.e f is continuous and possesses partial derivative at a point then f need not be differentiable at that point.

For example :-

$$f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2} ; & (x,y) \neq (0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$$ is continuous and possesses partial derivatives but is not differentiable at the origin.

Sol put $x = r\cos\theta$, $y = r\sin\theta$

$$\therefore \left| \frac{x^3-y^3}{x^2+y^2} - 0 \right| = \left| r(\cos^3\theta - \sin^3\theta) \right|$$

$$\leq |r| \left[|\cos^3\theta| + |\sin^3\theta| \right]$$

$$\leq |r| [1+1]$$

$$= 2|r|.$$

$$= 2\sqrt{x^2+y^2} < \epsilon$$

whenever $x^2 < \frac{\epsilon^2}{8}$, $y^2 < \frac{\epsilon^2}{8}$

(or)

$|x| < \frac{\epsilon}{2\sqrt{2}}$, $|y| < \frac{\epsilon}{2\sqrt{2}}$

$\therefore \left| \frac{x^3-y^3}{x^2+y^2} - 0 \right| < \epsilon$ whenever $|x| < \frac{\epsilon}{2\sqrt{2}}$, $|y| < \frac{\epsilon}{2\sqrt{2}}$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0$$

$\therefore f(x,y)$ is conti. at $(0,0)$

$$\lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{h^3}{h^2}-0}{h} = \lim_{h\to 0} \frac{h^3}{h^3} = 1$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k)-f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{\frac{-k^3}{k^2}-0}{k} = \lim_{k\to 0} \frac{-k^3}{k^3} = -1$$

$\therefore$ f possesses partial derivatives at $(0,0)$.

Now we prove that f is not differentiable at $(0,0)$.

If possible suppose that f is differentiable at $(0,0)$.

Then $f(0+h,0+k) - f(0,0) = h f_x(0,0) + k f_y(0,0) + \sqrt{h^2+k^2}\,\phi(h,k)$

where $\phi(h,k) \to 0$ as $(h,k) \to (0,0)$.

where $A = f_x(0,0) = 1$
and $B = f_y(0,0) = -1$.

$\Rightarrow f(h,k) = h - k + \sqrt{h^2+k^2}\,\phi(h,k)$

$$\Rightarrow \phi(h,k) = \frac{f(h,k)-h+k}{\sqrt{h^2+k^2}}$$ where $\phi(h,k) \to 0$ as $(h,k) \to (0,0)$

i.e., $$\lim_{(h,k)\to(0,0)} \phi(h,k) = \lim_{(h,k)\to(0,0)} \frac{f(h,k)-h+k}{\sqrt{h^2+k^2}} = 0$$

Now if $h = r\cos\theta$, $k = r\sin\theta$

then

$$\frac{f(h,k)-h+k}{\sqrt{h^2+k^2}} = \frac{\frac{r^3\cos^3\theta - r^3\sin^3\theta}{r^2(\cos^2\theta+\sin^2\theta)} - r\cos\theta + r\sin\theta}{\sqrt{r^2(\cos^2\theta+\sin^2\theta)}}$$

$$= \cos^3\theta - \sin^3\theta - \cos\theta + \sin\theta$$

$$\therefore 0 = \lim_{(h,k)\to(0,0)} \frac{f(h,k) - h + k}{\sqrt{h^2+k^2}} = \lim_{r\to 0} \left(\cos^3\theta - \sin^3\theta - \cos\theta + \sin\theta\right) \quad (1)$$

Since the expression $\cos^3\theta - \sin^3\theta - \cos\theta + \sin\theta$ is independent of r and (1) implies that

$\cos^3\theta - \sin^3\theta - \cos\theta + \sin\theta = 0 \quad \forall\, \theta$

which is impossible for arbitrary θ

$\therefore$ our assumption that f is differentiable is wrong.

$\therefore$ f is not differentiable at $(0,0)$

Sufficient condition for differentiability:

Theorem: If (a,b) be a point of the domain of definition of a function f such that

(i) f_x is continuous at (a,b)

(ii) f_y exists at (a,b)

then f is differentiable at (a,b).

Similarly, the statement that f is differentiable at (a,b) if f_x exists at (a,b) and f_y is continuous at (a,b) is true.

i.e., the continuity of one of partial derivatives and the existence of other guarantees the differentiability of the function under consideration.

Note: The conditions of the theorem are not necessary for differentiability. i.e., a function can be differentiable at a point even when none of the partial derivatives is continuous at that point.

However if the function is not differentiable at a point the partial derivatives cannot be continuous there.

For example:

(i) Consider the function $f: \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} x^2 \sin\frac{1}{x} + y^2 \sin\frac{1}{y} & ; \text{ if } xy \neq 0 \\ x^2 \sin\frac{1}{x} & ; \text{ if } x \neq 0 \text{ and } y = 0 \\ y^2 \sin\frac{1}{y} & ; \text{ if } x = 0 \text{ and } y \neq 0 \\ 0, & \text{ if } x = y = 0. \end{cases}$$

Prove that f is differentiable at $(0,0)$ but neither f_x nor f_y is continuous at $(0,0)$.

Sol^n: Here the partial derivatives at $(0,0)$ are given by

$$f_x(x,y) = \begin{cases} 2x \sin\frac{1}{x} - \cos\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

and $$f_y(x,y) = \begin{cases} 2y \sin\frac{1}{y} - \cos\frac{1}{y} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

Since $\underset{x \to 0}{Lt} \cos\frac{1}{x}$ doesnot exists and $\underset{t \to 0}{Lt}\, t \sin\frac{1}{t} = 0$.

$\therefore \underset{(x,y)\to(0,0)}{Lt} f_x(x,y)$ & $\underset{(x,y)\to(0,0)}{Lt} f_y(x,y)$ do not exist.

i.e, f_x and f_y are discontinuous at $(0,0)$.

$\therefore$ Both the partial derivatives exist at $(0,0)$ but neither f_x nor f_y is continuous at $(0,0)$.

Now show that the function is differentiable at $(0,0)$.

Here
$$f(h,k) - f(0,0) = h^2 \sin\frac{1}{h} + k^2 \sin\frac{1}{k}$$
$$= 0h + 0k + h \cdot h \sin\frac{1}{h} + k \cdot k \sin\frac{1}{k}$$

Now $\underset{(h,k)\to(0,0)}{Lt} h \sin\frac{1}{h} = 0$ and $\underset{(h,k)\to(0,0)}{Lt} k \sin\frac{1}{k} = 0$

$\therefore f$ is differentiable at $(0,0)$.

Note:

→ A real valued function f of two variables is said to be continuously at a point (a,b) if both the first order partial derivatives exist in a nbd of (a,b) and are continuous at the point (a,b).

→ A function, which is continuously differentiable at a point is differentiable at that point.

→ Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function given by

$$f(x,y) = \begin{cases} xy\,\dfrac{x^2-y^2}{x^2+y^2}, & \text{if } x^2+y^2 \neq 0 \\ 0, & \text{if } x=y=0 \end{cases}$$

Show that f is differentiable at $(0,0)$.

Sol:

Now $f_x(0,0) = \lim_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h}$

$= 0$

Similarly $f_y(0,0) = 0$

and for $(x,y) \neq (0,0)$

$$f_x(x,y) = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

Using polar co-ordinates

$x = r\cos\theta$, $y = r\sin\theta$

we get $|f_x(x,y) - f_x(0,0)| = r\,|\cos^4\theta\sin\theta + 4\cos^2\theta\sin^3\theta - \sin^5\theta|$

$\leq r\,[|\cos^4\theta\sin\theta| + 4|\cos^2\theta\sin^3\theta| + |\sin^5\theta|]$

$= 6r$ $(\because \sin\theta \leq 1$ & $\cos\theta \leq 1)$

$= 6\sqrt{x^2+y^2}$.

$< \epsilon$ if $|x| < \dfrac{\epsilon}{\sqrt{72}}$ and $|y| < \dfrac{\epsilon}{\sqrt{72}}$.

$\therefore \lim_{(x,y)\to(0,0)} f_x(x,y) = f_x(0,0)$

$\therefore$ we have f_x is continuous at $(0,0)$ and f_y exists at $(0,0)$

$\therefore$ f is differentiable at $(0,0)$.

→ S.T $f(x,y) = e^{x+y}\sin x + x^2 + 2xy$ is differentiable everywhere.

Sol: Since $f_x(x,y) = e^{x+y}\sin x + e^{x+y}\cos x + 2x + 2y$ and $f_y(x,y) = e^{x+y}\sin x + 2x$ ... everywhere ... differentiable everywhere

PARTIAL DERIVATIVES OF HIGHER ORDER

We studied partial derivatives of first order and differentiability. We must have seen often, that partial derivatives of first order again define functions.

For example:

if $f(x,y) = 3x^3 + 2xy^2 + 5y^2 + 6$.

then $f_x(x,y) = 9x^2 + 2y^2$

and $f_y(x,y) = 4xy + 10y$ are again real valued functions of two variables with the domain $\mathbb{R}^2$.

Thus we can talk of first order partial derivatives of these new functions.

If we consider a function of two variables, there are two first order partial derivatives, which may give rise to four more partial derivatives, which might again turn out to be function. If this chain continues, then we obtain Higher order partial derivatives.

In general, let $D \subset \mathbb{R}^2$ and let $f: D \to \mathbb{R}$ have a first order partial derivative $f_x(x,y)$ at every point of D. This new function f_x, which is defined on D may or may not possess first order partial derivatives.

In case it does, then f_{xx} and f_{xy} are called

the second order partial derivatives of f.

Similarly, if the function f has a first order partial derivative f_y at every point of D, then f_y defines a new function and if this new function has first order partial derivatives, then we get two more second order partial derivatives, namely, f_{xy} and f_{yy}.

$\therefore$ If $f(x,y)$ is a real-valued function defined in a nbd of (a,b) having both the partial derivatives at all the points of the nbd, then

$$f_{xx}(a,b) = \lim_{h \to 0} \frac{f_x(a+h,b) - f_x(a,b)}{h} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \text{(say)}$$

$$f_{yx}(a,b) = \lim_{k \to 0} \frac{f_x(a,b+k) - f_x(a,b)}{k} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \text{(say)}$$

$$f_{xy}(a,b) = \lim_{h \to 0} \frac{f_y(a+h,b) - f_y(a,b)}{h} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \text{(say)}$$

$$f_{yy}(a,b) = \lim_{k \to 0} \frac{f_y(a,b+k) - f_y(a,b)}{k} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \text{(say)}$$

provided each one of these limits exists.

The second order partial derivatives of f are also denoted by

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}; \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y};$$

$$f_{yx} = \frac{\partial^2 f}{\partial y \partial x}; \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

If we want to indicate the particular point at which the second order partial derivatives are taken, then we write

$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)}$, $\frac{\partial^2 f(a,b)}{\partial x^2}$, $f_{xx}(a,b)$ or $f_{x^2}(a,b)$;

$\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(a,b)}$; $\frac{\partial^2 f(a,b)}{\partial x \partial y}$ or $f_{xy}(a,b)$ and so on.

In a similar manner partial derivatives of order higher than two are defined.

for example

$$\frac{\partial^3 f}{\partial x \partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right]$$

$$= f_{xxy} \text{ or } f_{x^2y}$$

i.e $\frac{\partial^3 f}{\partial x \partial x \partial y}$ stands for the partial derivative of $\frac{\partial^2 f}{\partial x \partial y}$ with respect to x.

→ Find all the second order partial derivatives of the following function.

(i) $f(x,y) = x^3 + y^3 + 3axy$; 'a' is constant

(ii) $f(x,y,z) = x^2 + yz + xz^3$.

Sol (i): $f(x,y) = x^3 + y^3 + 3axy$

$\frac{\partial f}{\partial x} = 3x^2 + 3ay$ and $\frac{\partial f}{\partial y} = 3y^2 + 3ax$

$$\therefore \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(3x^2 + 3ay) = 6x$$

$$\text{and } \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(3x^2 + 3ay) = 3a$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(3y^2 + 3ax) = 3a$$

$$\text{and } \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 6y.$$

(ii) For $f(x,y,z) = x^2 + yz + xz^3$.

$$\frac{\partial f}{\partial x} = 2x + z^3 \; ; \; \frac{\partial f}{\partial y} = z \; ; \; \frac{\partial f}{\partial z} = y + 3xz^2.$$

$$\therefore \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 2 \; ; \; \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 0 \; ;$$

$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = 3z^2.$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 0 \; ; \; \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 0$$

$$\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = 1$$

$$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) = 3z^2 \; ; \; \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = 1$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = 6xz.$$

→ If $f(x,y) = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$, $x \neq 0, y \neq 0$

Show that $\frac{\partial^2 f}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$.

Sol^n: $f(x,y) = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$

Now, $$\frac{\partial f}{\partial y} = x^2 \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} - 2y \tan^{-1}\frac{x}{y} - y^2 \cdot \frac{1}{1+\frac{x^2}{y^2}} \cdot \left(-\frac{x}{y^2}\right)$$

$$= \frac{x^3}{x^2+y^2} - 2y \tan^{-1}\frac{x}{y} + \frac{xy^2}{x^2+y^2}$$

$$= \frac{x(x^2+y^2)}{(x^2+y^2)} - 2y \tan^{-1}\frac{x}{y}$$

$$= x - 2y \tan^{-1}\frac{x}{y}$$

$$\therefore \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(x - 2y \tan^{-1}\frac{x}{y}\right)$$

$$= 1 - 2y \frac{1}{1+\frac{x^2}{y^2}} \cdot \left(\frac{1}{y}\right)$$

$$= 1 - \frac{2y^2}{x^2+y^2}$$

$$= \frac{x^2-y^2}{x^2+y^2}$$

HW → If $f(x,y,z) = e^{xyz}$, then show that

$$\frac{\partial^3 f}{\partial x \partial y \partial z} = \left(1 + 3xyz + x^2y^2z^2\right) e^{xyz}.$$

→ Find all the second order partial derivatives of the following functions:

(a) $f(x,y) = \cos\frac{y}{x}$ (b) $f(x,y) = x^5 + y^4 \sin x^6$

(c) $f(x,y,z) = \sin xy + \tan yz + \cos xz$

(d) $f(x,y,z) = xyz^2 + xyz + x^3y$

→ If $f(x,y,z) = (x^2+y^2+z^2)^{-1/2}$.

Show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$

→ verify that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for each of the following functions

(a) $f(x,y) = x^3y + e^{xy^2}$ (b) $f(x,y) = \tan(xy^3)$

(c) $f(x,y) = \frac{2x-y}{x+y}$ (d) $f(x,y) = x \tan xy$

→ Already we have seen that it is not always possible to find first order partial derivatives by direct differentiation.
The same is true for higher order partial derivatives of some functions.

(1) For example:
Consider the function

$$f(x,y) = \begin{cases} xy\dfrac{(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

Find the second order partial derivatives of f at $(0,0)$.

Soln: Since $f_{xx}(0,0) = \lim_{h \to 0} \dfrac{f_x(h,0) - f_x(0,0)}{h}$ ——(1)

we first evaluate $f_x(h,0)$ and $f_x(0,0)$.

Now $f_x(0,0) = \lim_{t \to 0} \dfrac{f(t,0) - f(0,0)}{t}$

$= \lim_{t \to 0} \dfrac{0-0}{t} = 0$

$f_x(h,0) = \lim_{t \to 0} \dfrac{f(h+t,0) - f(h,0)}{t}$

$= \lim_{t \to 0} \dfrac{0-0}{t} = 0$ & $f_x(h,0)$ in (1)

Substitute the values of $f_x(0,0)$

$\therefore f_{xx}(0,0) = \lim_{h \to 0} \dfrac{0-0}{h} = 0$

Since $f_{xy}(0,0) = \lim_{h \to 0} \dfrac{f_y(h,0) - f_y(0,0)}{h}$ ——(2)

now let $f_y(h,0) = \lim_{t \to 0} \dfrac{f(h,t) - f(h,0)}{t}$

$$\therefore \lim_{t\to 0} \frac{f(h,t)-f(h,0)}{t}$$

$$= \lim_{t\to 0} \frac{h\cancel{t}(h^2-t^2)}{\cancel{t}(h^2+t^2)} = h$$

Now let $f_y(0,0) = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t}$

$$= \lim_{t\to 0} \frac{0-0}{t} = 0$$

$$\therefore f_{xy}(0,0) = \lim_{h\to 0} \frac{h-0}{h} = 1$$

Since $f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,0+k)-f_x(0,0)}{k}$

now let $f_x(0,k) = \lim_{t\to 0} \frac{f(t,k)-f(0,k)}{t}$

$$= \lim_{t\to 0} \frac{\frac{tk(t^2-k^2)}{t^2+k^2} - 0}{t}$$

$$= \lim_{t\to 0} \frac{k(t^2-k^2)}{t^2+k^2} = -k$$

and $f_x(0,0) = 0$

$$\therefore f_{yx}(0,0) = \lim_{k\to 0} \frac{-k-0}{k} = -1$$

Since $f_{yy}(0,0) = \lim_{k\to 0} \frac{f_y(0,k)-f_y(0,0)}{k}$

now let $f_y(0,k) = \lim_{t\to 0} \frac{f(0,k+t)-f(0,k)}{t}$

$$= \lim_{t\to 0} \frac{0-0}{t} = 0$$

and $f_y(0,0) = 0$

$$\therefore f_{yy}(0,0) = \lim_{k\to 0} \frac{0-0}{t} = 0$$

H.W. → Evaluate $f_{xy}(0,0)$ and $f_{yx}(0,0)$, for the function given by

$$f(x,y) = \begin{cases} (x^4+y^4)\tan^{-1}\left(\frac{y^2}{x^2}\right), & x \neq 0 \\ \frac{\pi y^4}{2}, & x = 0 \end{cases}$$

Note: Now we will given an example of a function whose first order partial derivatives exist, but higher order ones do not exist and also we see that the existence of a partial derivative of a particular order does not imply the existence of other partial derivatives of the same order.

(II) **For example:**

Consider the function

$$f(x,y) = \begin{cases} \frac{xy^2}{\sqrt{x^2+y^2}}, & xy \neq 0 \\ 0, & xy = 0 \end{cases}$$

Examine whether the second order partial derivatives of f at $(0,0)$ exist or not.

Sol^n: Now $f_{xx}(0,0) = \lim_{h\to 0} \frac{f_x(0+h,0) - f_x(0,0)}{h}$

let $f_x(h,0) = \lim_{t\to 0} \frac{f(h+t,0) - f(h,0)}{t}$

$$= \lim_{t\to 0} \frac{0-0}{t} = 0$$

and $f_x(0,0) = \lim_{t\to 0} \frac{f(0+t,0) - f(0,0)}{t}$

$$= \lim_{t\to 0} \frac{0-0}{t} = 0$$

$\therefore f_{xx}(0,0) = \lim_{h\to 0} \frac{0-0}{h} = 0$

Now $f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,0+k) - f_x(0,0)}{k}$

let $f_x(0,k) = \lim_{t\to 0} \frac{f(t,k) - f(0,k)}{t}$

$$= \lim_{t\to 0} \frac{\frac{tk^2}{\sqrt{t^2+k^2}} - 0}{t}$$

$$= \lim_{t\to 0} \frac{k^2}{\sqrt{t^2+k^2}} = \frac{k^2}{|k|} = \pm k \text{ (or) } |k|$$

and $f_x(0,0) = 0$

Now,

$$f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} = \lim_{k\to 0} \frac{|k|}{k}$$

which doesnot exist.

$\therefore$ f_{yx} doesnot exist at $(0,0)$

Now

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

$\because$ Let $f_y(h,0) = \lim_{t\to 0} \frac{f(h,t) - f(h,0)}{t}$

$$= \lim_{t\to 0} \frac{\frac{ht^2}{\sqrt{h^2+t^2}} - 0}{t}$$

$$= \lim_{t\to 0} \frac{ht}{\sqrt{h^2+t^2}} = 0$$

and $f_y(0,0) = \lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t}$

$$= \lim_{t\to 0} \frac{0-0}{t} = 0$$

$$\therefore f_{xy}(0,0) = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

Now $f_{yy}(0,0) = \lim_{k\to 0} \frac{f_y(0,k) - f_y(0,0)}{k}$

let $f_y(0,k) = \lim_{t\to 0} \frac{f(0,k+t) - f(0,k)}{t}$

$$= \lim_{t\to 0} \frac{0-0}{t} = 0$$

and $f_y(0,0) = 0$.

$$\therefore f_{yy}(0,0) = \lim_{k\to 0} \frac{0-0}{k} = 0$$

$\therefore$ f_{xx}, f_{xy} and f_{yy} exist at $(0,0)$ and are equal to zero, while $f_{yx}(0,0)$ doesnot exist

The study of above examples must have convinced that we have to be careful about the order of variables w.r.t which higher order derivatives are taken. For instance, from example ⑰ it is clear that f_{xy} need not be equal to f_{yx}.

Example ⑱ goes a step further, where f_{yx} exist at (0,0), while f_{xy} does not, showing that the question of their equality does not arise at all.

If we look at the definitions of f_{xy} and f_{yx} at a point (a,b) more carefully, we would see why the expectation of the equality $f_{xy}(a,b) = f_{yx}(a,b)$ is farfetched (very difficult to believe)

Now by definition

$$f_{xy}(a,b) = \lim_{h\to 0} \frac{f_y(a+h,b) - f_y(a,b)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\lim_{k\to 0} \frac{f(a+h,b+k)-f(a+h,b)}{k} - \lim_{k\to 0} \frac{f(a,b+k)-f(a,b)}{k}\right]$$

$$= \lim_{h\to 0} \lim_{k\to 0} \frac{f(a+h,b+k)-f(a+h,b)-f(a,b+k)+f(a,b)}{hk}$$

$$= \lim_{h\to 0} \lim_{k\to 0} \frac{\phi(h,k)}{hk}$$

where $\phi(h,k) = f(a+h,b+k) - f(a+h,b) - f(a,b+k) + f(a,b)$

Similarly

$$f_{yx}(a,b) = \lim_{k\to 0} \lim_{h\to 0} \frac{\phi(h,k)}{hk}$$

The expressions $f_{xy}(a,b)$ and $f_{yx}(a,b)$ are the repeated limits of the same expression taken in different orders. and we have already seen that repeated limits are not equal, in general.

Now we will give the conditions under which these mixed partial derivatives become equal.

Sufficient conditions for the equality of f_{xy} and f_{yx}:

Theorem (I): Let $f(x,y)$ be a real valued function such that the two second order partial derivatives f_{xy} and f_{yx} are continuous at a point (a,b).

Then $f_{xy}(a,b) = f_{yx}(a,b)$

Schwarz's theorem:

Let $f(x,y)$ be a real valued function defined in a nbd of (a,b) such that in a certain nbd. of (a,b)

(i) f_x exists

(ii) f_{xy} is continuous at (a,b).

Then f_{yx} exists at (a,b)

and $f_{yx}(a,b) = f_{xy}(a,b)$.

Note: The conditions in Schwarz's theorem are less restrictive than those in theorem (I).

→ Evaluate f_{xy} at a point (x,y) for the function f defined by $f(x,y) = x^4 + x^2y^2 + y^6$. Use Schwarz's theorem to evaluate f_{yx} at the point (x,y).

Sol: By direct differentiation

$f_y(x,y) = 2x^2y + 6y^5$

$\Rightarrow f_{xy}(x,y) = 4xy$.

Since $4xy$ is a polynomial.

$\therefore f_{xy} = 4xy$ is a continuous function.

Further $f_x = 4x^3 + 2xy^2$ exist.

$\therefore f$ satisfies the conditions of Schwarz's theorem.

$\therefore f_{xy} = f_{yx} = 4xy$

Note: In theorem (I) we assume that both the mixed partial derivatives are continuous, whereas in Schwarz's theorem we assume that only one of them, say f_{xy} is continuous and that f_x exists.

But even though the conditions of Schwarz's theorem are less strict, these are still not necessary for the equality of mixed partial derivatives.

In otherwords, we can have functions whose mixed partial derivatives at some point are equal, but which donot satisfy the requirements of Schwarz's theorem.

For example:

Consider the function f defined by—

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & x = y = 0. \end{cases}$$

Show that $f_{xy}(0,0) = f_{yx}(0,0)$; even though f doesnot fulfill the requirements of Schwarz's theorem.

Sol^n: Since $f_{xy}(0,0) = \lim_{h\to 0} \dfrac{f_y(h,0) - f_y(0,0)}{h}$

Now, $f_y(h,0) = \lim_{t\to 0} \dfrac{f(h,t) - f(h,0)}{t}$

$$= \lim_{t\to 0} \frac{\frac{h^2t^2}{h^2+t^2} - 0}{t}$$

$$= \lim_{t\to 0} \frac{h^2t}{h^2+t^2} = 0.$$

and $f_y(0,0) = \lim_{t\to 0} \dfrac{f(0,t) - f(0,0)}{t}$

$$= \lim_{t\to 0} \frac{0-0}{t} = 0$$

$\therefore f_{xy}(0,0) = \lim_{h\to 0} \frac{0-0}{h} = 0.$

Since $f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$

Now let $f_x(0,k) = \lim_{t\to 0} \frac{f(t,k) - f(0,k)}{t}$

$$= \lim_{t\to 0} \frac{\frac{t^2k^2}{t^2+k^2} - 0}{t}$$

$$= \lim_{t\to 0} \frac{tk^2}{t^2+k^2} = 0$$

and $f_x(0,0) = \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t}$

$$= \lim_{t\to 0} \frac{0-0}{t} = 0$$

$\therefore f_{yx}(0,0) = 0.$

$\therefore f_{xy}(0,0) = f_{yx}(0,0).$

Now we show that the conditions of Schwarz's theorem are not satisfied.

For $(x,y) \neq (0,0)$,

we can find the partial derivatives of f at (x,y) by differentiating directly.

$\therefore f_y(x,y) = \frac{\partial}{\partial y}\left(\frac{x^2y^2}{x^2+y^2}\right)$

$$= \frac{2(x^2+y^2)x^2y - 2x^2y^3}{(x^2+y^2)^2}$$

$$= \frac{2x^4y}{(x^2+y^2)^2}$$

Further $f_{xy} = \frac{\partial}{\partial x}\left[\frac{2x^4y}{(x^2+y^2)^2}\right]$

$$= \frac{8x^3y(x^2+y^2)^2 - 8x^5y(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{(8x^3y(x^2+y^2) - 8x^5y)(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{8x^3y(x^2+y^2-x^2)}{(x^2+y^2)^3}$$

$$= \frac{8x^3y^3}{(x^2+y^2)^3}.$$

$$\lim_{(x,y)\to(0,0)} \frac{8x^3y^3}{(x^2+y^2)^3} \text{ doesnot exist.}$$

for, if (x,y) approach $(0,0)$ along the line $y=mx$.

we get

$$\lim_{(x,y)\to(0,0)} \frac{8x^3y^3}{(x^2+y^2)^3} = \lim_{x\to 0} \frac{8 \cdot x^3 m^3 x^3}{(x^2+m^2x^2)^3}$$

$$= \lim_{x\to 0} \frac{8m^3}{(1+m^2)^3}$$

The limit is different for different values of m. i.e, the limit doesnot exist

$$\therefore \lim_{(x,y)\to(0,0)} f_{xy}(x,y) \neq f_{xy}(0,0) = 0.$$

which implies that f_{xy} is not continuous at $(0,0)$.

Young's theorem:

Let $f(x,y)$ be a real-valued function defined in a nbd of a point (a,b) such that both the first order partial derivatives f_x and f_y are differentiable at (a,b).

Then $f_{xy}(a,b) = f_{yx}(a,b)$.

Note: As in the case of Schwarz's theorem the conditions stated in young's theorem are less strict than in theorem (I).

However, these are not necessary for the equality of mixed partial derivatives.

→ Consider the function f defined by

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & x=y=0 \end{cases}$$

S.T. $f_{xy}(0,0) = f_{yx}(0,0)$, even though the conditions of Young's theorem are not satisfied.

Sol: Already we have seen that

$$f_{xy}(0,0) = f_{yx}(0,0) = 0$$

Let us now show that the conditions of Young's theorem are not satisfied.

Now we prove that f_x is not differentiable at $(0,0)$.

For this, assume that f_x is differentiable at $(0,0)$.

Then there exist functions $\phi(h,k)$ and $\psi(h,k)$ such that

$$f_x(h,k) - f_x(0,0) = h f_{xx}(0,0) + k f_{yx}(0,0) + h\,\phi(h,k) + k\,\psi(h,k)$$

and $\phi(h,k) \to 0$ as $(h,k) \to (0,0)$;

$\psi(h,k) \to 0$ as $(h,k) \to (0,0)$.

Now let $f_{xx}(0,0) = \lim_{h\to 0} \dfrac{f_x(h,0) - f_x(0,0)}{h}$ —(2)

Let $f_x(h,0) = \lim_{t\to 0} \dfrac{f(h+t,0) - f(h,0)}{t}$

$= \lim_{t\to 0} \dfrac{0-0}{t} = 0$

and $f_x(0,0) = \lim_{t\to 0} \dfrac{f(t,0) - f(0,0)}{t}$

$= \lim_{t\to 0} \dfrac{0-0}{t} = 0$

(2) ≡ $f_{xx}(0,0) = \lim_{h\to 0} \frac{0-0}{h} = 0$

∴ eqn (1) becomes

$f_x(h,k) - 0 = h\,\phi(h,k) + k\,\psi(h,k)$ ——— (3)

$(\because f_{yx}(0,0) = 0 \ \&\ f_{xx}(0,0) = 0)$

for $(x,y) \neq (0,0)$

$f_x(x,y) = \frac{2xy^2(x^2+y^2) - 2x(x^2y^2)}{(x^2+y^2)^2}$

$= \frac{2xy^4}{(x^2+y^2)^2}$

$\Rightarrow f_x(h,k) = \frac{2hk^4}{(h^2+k^2)^2}$

∴ (3) ≡

$\frac{2hk^4}{(h^2+k^2)^2} = h\,\phi(h,k) + k\,\psi(h,k)$.

putting $h = r\cos\theta$, $k = r\sin\theta$.

we get

$\frac{2r^5\cos\theta\sin^4\theta}{r^4} = r\cos\theta\,\phi(r\cos\theta, r\sin\theta) + r\sin\theta\,\psi(r\cos\theta, r\sin\theta)$

$2\cos\theta\sin^4\theta = \cos\theta\,\phi(r\cos\theta, r\sin\theta) + r\sin\theta\,\psi(r\cos\theta, r\sin\theta)$ ——— (4)

for arbitrary θ, $(h,k) = (r\cos\theta, r\sin\theta) \to (0,0)$ as $r \to 0$.

and $\because \phi(h,k) \to 0$ and $\psi(h,k) \to 0$.

Taking the limit of (4) as $r \to 0$, we get

$2\cos\theta\sin^4\theta = 0$.

which is impossible for arbitrary θ.

∴ f_x is not differentiable at $(0,0)$.

Similarly, we can show that f_y is not differentiable at $(0,0)$.

∴ The function f doesnot satisfy the conditions of young's theorem, even though we have $f_{xy}(0,0) = f_{yx}(0,0)$

Differentials of higher order :

Let $z = f(x, y)$ be a function of two independent variables x and y, defined in a domain N and let it be differentiable at a point (x, y) of the domain. The first differential of z at (x, y), denoted by dz is given by

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \quad \text{--- (1)}$$

— If dx and dy are regarded as constants and if $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are differentiable at (x, y) then dz is a function of x and y and itself is differentiable at (x,y), then the differential of dz, called the second differential of z is denoted by d^2z.

$$\therefore d^2z = d(dz)$$

$$= d\left(\frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy\right)$$

$$= d\left(\frac{\partial z}{\partial x}\right) dx + d\left(\frac{\partial z}{\partial y}\right) dy \quad \text{--- (2)} \qquad (\because dx \,\&\, dy \text{ are constants})$$

Now let $d\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) dx + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) dy$ (from (1))

$$= \frac{\partial^2 z}{\partial x^2} dx + \frac{\partial^2 z}{\partial y \partial x} dy$$

Similarly $d\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} dx + \frac{\partial^2 z}{\partial y^2} dy$ --- (3)

Also by young's theorem, since $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are differentiable.

we have $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ --- (4)

from (1), (2) & (4)

$$d^2z = \frac{\partial^2 z}{\partial x^2} dx^2 + 2\frac{\partial^2 z}{\partial x \partial y} dx\, dy + \frac{\partial^2 z}{\partial y^2} dy^2 \quad \text{--- (5)}$$

where $dx^2 = dx \cdot dx = (dx)^2$

$dy^2 = (dy)^2$.

Eqn (5), can be written as

$$d^2z = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^2 z \quad \text{——— (6)}$$

Again d^2z is differentiable at (x,y) if all the second order partial derivatives $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y^2}$ are differentiable at (x,y).

$$\therefore d^3z = d(d^2z) = d\left[\frac{\partial^2 z}{\partial x^2}dx^2 + 2\frac{\partial^2 z}{\partial x \partial y}dx\,dy + \frac{\partial^2 z}{\partial y^2}dy^2\right]$$

$$= d\left(\frac{\partial^2 z}{\partial x^2}\right)dx^2 + 2\,d\left(\frac{\partial^2 z}{\partial x \partial y}\right)dx\,dy + d\left(\frac{\partial^2 z}{\partial y^2}\right)dy^2$$

$$= \left[\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial x^2}\right)dy\right]dx^2$$

$$+ 2\left[\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x \partial y}\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial x \partial y}\right)dy\right]dx\,dy$$

$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y^2}\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial y^2}\right)dy\right]dy^2$$

$$= \frac{\partial^3 z}{\partial x^3}dx^3 + \frac{\partial^3 z}{\partial y \partial x^2}dy\,dx^2 + 2\frac{\partial^3 z}{\partial x^2 \partial y}dx^2 dy$$

$$+ 2\frac{\partial^3 z}{\partial y\,\partial x \partial y}dy^2 dx + \frac{\partial^3 z}{\partial x \partial y^2}dx\,dy^2 + \frac{\partial^3 z}{\partial y^3}dy^3$$

$$= \frac{\partial^3 z}{\partial x^3}dx^3 + 3\frac{\partial^3 z}{\partial x^2 \partial y}dx^2 dy + 3\frac{\partial^3 z}{\partial x \partial y^2}dx\,dy^2 + \frac{\partial^3 z}{\partial y^3}dy^3$$

$(\because z_{xy} = z_{yx},\ z_{xyy} = z_{yyx})$

$$= \left[\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right]^3 z.$$

In general, $d^n z = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^n z$ exists if $d^{n-1}z$ is differentiable.

Note: In the above discussion, x & y are independent variables and so dx and dy may be treated as constants. The reason for this being so is that the differentials of independent variables are the arbitrary increments of these variables, $dx = \delta x$, $dy = \delta y$.

Functions of functions:

So far we have considered functions of the form $z = f(x, y, ----)$ where the variables $x, y, ---$ are the independent variables.

Now we consider functions

$z = f(x, y, \ldots)$

where $x, y, \ldots$ are not independent variables but are themselves functions of other independent variables $u, v, \ldots$, so that

$x = g(u, v, \ldots)$ and $y = h(u, v, \ldots)$

Theorem:
If $z = f(x, y)$ is a differentiable function of x, y and $x = g(u, v)$, $y = h(u, v)$ are themselves differential functions of the independent variables, u, v, then z is a differentiable function of u, v and

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy.$$

just as though x, y are the independent variables

Note: (1) The theorem establishes a fact of fundamental importance that the first differential of a function is expressed always by the same formula, whether the variables concerned are independent or whether they are themselves functions of other independent variables.

(2) The differential dz is sometimes referred to as the total differential.

Differentials of Higher order of a function of functions.

If $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are differentiable functions of x, y. so that they are also differentiable functions of

u, v and dx, dy are differentiable functions of u, v; then from the above theorem we have

$$d^2z = d(dz)$$
$$= d\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right) \qquad \left(\because dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right)$$
$$= d\left(\frac{\partial z}{\partial x}\right)dx + \frac{\partial z}{\partial x}d(dx) + d\left(\frac{\partial z}{\partial y}\right)dy + \frac{\partial z}{\partial y}d(dy)$$
$$= \left[\frac{\partial^2 z}{\partial x^2}dx + \frac{\partial^2 z}{\partial y \partial x}dy\right]dx + \frac{\partial z}{\partial x}d^2x + \left[\frac{\partial^2 z}{\partial x \partial y}dx + \frac{\partial^2 z}{\partial y^2}dy\right]dy + \frac{\partial z}{\partial y}d^2y.$$

$$d^2z = \frac{\partial^2 z}{\partial x^2}dx^2 + 2\frac{\partial^2 z}{\partial x \partial y}dx\,dy + \frac{\partial^2 z}{\partial y^2}dy^2 + \frac{\partial z}{\partial x}d^2x + \frac{\partial z}{\partial y}d^2y.$$
$$= \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^2 z + \frac{\partial z}{\partial x}d^2x + \frac{\partial z}{\partial y}d^2y.$$

→ The differentials of higher orders can be written in the same manner, but their formation becomes more and more complicated and lengthy, and no simple general formula for $d^n z$ can be given.

→ The introduction of more than two variables, which are functions of independent variables causes no difficulty.

Thus, when $z = f(x_1, x_2, x_3)$, and x_1, x_2, x_3 are not the independent variables,

$$d^2z = \left(\frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2}dx_2 + \frac{\partial}{\partial x_3}dx_3\right)^2 z + \frac{\partial z}{\partial x_1}d^2x_1 + \frac{\partial z}{\partial x_2}d^2x_2 + \frac{\partial z}{\partial x_3}d^2x_3.$$

Note: If x, y are linear functions of independent variables u and v, i.e., x and y of the form $x = a + bu + cv$, $y = a' + b'u + c'v$ then dx and dy are constants. and d^2x, d^2y and all higher differentials of x and y are zero, and

$$\therefore d^n z = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^n z$$

the form being same as for independent x & y.

→ The Derivation of composite Functions (The chain Rule):

Let $Z = f(x,y)$ possess continuous first order partial derivatives.
Let $x = \phi(t)$, $y = \psi(t)$ possess continuous derivatives
Then the composite function
$Z = f(\phi(t), \psi(t))$ has derivative given by

$$\frac{dZ}{dt} = \frac{\partial Z}{\partial x}\frac{dx}{dt} + \frac{\partial Z}{\partial y}\frac{dy}{dt}$$

($\frac{dZ}{dt}$ is called the total derivative)

Because:

since x, y are differentiable functions of single variable 't'

$\therefore$ $dx = \frac{dx}{dt}.dt$ and $dy = \frac{dy}{dt}.dt$

Since Z is differentiable function of x and y and x, y are differentiable functions of t.

$\therefore$ Z is a differentiable function of t.

$\therefore$ $dZ = \frac{dZ}{dt}.dt$ ——— (1)

Also $dZ = \frac{\partial Z}{\partial x}.dx + \frac{\partial Z}{\partial y}dy$

$\Rightarrow dZ = \frac{\partial Z}{\partial x}.\frac{dx}{dt}dt + \frac{\partial Z}{\partial y}\frac{dy}{dt}dt$ ——— (2)

from (1) & (2) we have

$$\boxed{\frac{dZ}{dt} = \frac{\partial Z}{\partial x}.\frac{dx}{dt} + \frac{\partial Z}{\partial y}\frac{dy}{dt}}$$

Corollary:— If $z = f(x,y)$ possesses 1st order partial derivatives, and x, y are linear functions of a single variable 't', i.e. $x = a + ht$, $y = b + kt$, where a, b, h, k are constants then

$$\frac{d^n z}{dt^n} = \left(h\,\partial/_{\partial x} + k\,\partial/_{\partial y}\right)^n z$$

Sol

Now $\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$

$= h\frac{\partial z}{\partial x} + k\frac{\partial z}{\partial y}$ — (1) where $\frac{dx}{dt} = h$, $\frac{dy}{dt} = k$

$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)z$ — (2)

Now $\frac{d^2 z}{dt^2} = \frac{d}{dt}\left(\frac{dz}{dt}\right)$

$= \frac{d}{dt}\left(h\frac{\partial z}{\partial x} + k\frac{\partial z}{\partial y}\right)$

$= h\frac{\partial}{\partial x}\left(h\frac{\partial z}{\partial x} + k\frac{\partial z}{\partial y}\right) + k\frac{\partial}{\partial y}\left(h\frac{\partial z}{\partial x} + k\frac{\partial z}{\partial y}\right)$

$= h^2\frac{\partial^2 z}{\partial x^2} + 2hk\frac{\partial^2 z}{\partial x\partial y} + k^2\frac{\partial^2 z}{\partial y^2}$

$= \left(h\frac{\partial}{\partial x} + k\,\partial/_{\partial y}\right)^2 z.$

$\therefore$ In general,

$$\frac{d^n z}{dt^n} = \left(h\,\partial/_{\partial x} + k\,\partial/_{\partial y}\right)^n z.$$

→ Let $z = f(x,y)$ be possess continuous first order partial derivatives.

Let $x = \phi(u,v)$
$y = \psi(u,v)$ possess continuous first order partial derivatives

Then $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$ and

$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$

* Derivative of an implicit function:

Let y be a function of x defined implicitely by the equation $f(x,y)=0$ — — by the above

$$\frac{\partial f}{\partial x}\frac{dx}{dx}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0$$

$$\Rightarrow f_x + f_y\frac{dy}{dx}=0$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{f_x}{f_y}}$$

provided $f_y \neq 0$.

Problems

Let $f(x,y) = ax^2+2hxy+by^2-1$ then find $\frac{dy}{dx}$.

Ans: $\frac{dy}{dx} = \frac{-f_x}{f_y} = -\frac{(ax+hy)}{hx+by}$

$\Rightarrow$ find $\frac{du}{dx}$ for $u=\sin(x^2-y^2)$ where x and y satisfy the eqn $a^2x^2+b^2y^2=c^2$.

Sol $\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx}+\frac{\partial u}{\partial y}\frac{dy}{dx}$

$= 2x\cos(x^2-y^2) + 2y\cos(x^2-y^2)\frac{dy}{dx}$

$= \cos(x^2-y^2)\left[2x+2y\frac{dy}{dx}\right]$ — (1)

Now let $\phi(x,y) = a^2x^2+b^2y^2-c^2=0$

then $\frac{dy}{dx} = -\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}}$

$= -\frac{2a^2x}{2b^2y}$

$\therefore$ (1) $\Rightarrow \frac{du}{dx} = \cos(x^2 - ey^2)\left[2x + 2y\left(-\frac{c^2 x}{b^2 y}\right)\right]$

$= 2x\cos(x^2+y^2)\left[1 - \frac{c^2}{b^2}\right]$

→ find $\frac{du}{dx}$, for each of the following problems.

(a) $u = x^2 - xy + y^2$, $y = 3x+2$

(b) $u = x^2 - y^3$, $y = \ln x$

(c) $u = x \ln xy$ where $x^3 + y^3 + 3xy = 1$.

*** Homogeneous functions :—**

A function $z = f(x,y)$ is called a homogeneous function of degree 'n' if it is expressible as $z = x^n g\left(\frac{y}{x}\right)$.

Ex :— $ax^2 + 2hxy + by^2 = x^2\left[a + 2h\left(\frac{y}{x}\right) + b\left(\frac{y}{x}\right)^2\right]$

$= x^2 g\left(\frac{y}{x}\right)$.

$\therefore$ It is a homo. function degree 2.

→ The following functions are homogeneous functions :

(i) $f(x,y) = \tan\left(\frac{y}{x}\right)$; degree '0'

(ii) $f(x,y) = \sqrt[3]{x^4 + 2y^4}$; degree $\frac{4}{3}$

(iii) $f(x,y) = \dfrac{\sin\left(\frac{x^2 y}{x^3+y^3}\right)}{\ln\left(\frac{x+y}{x}\right)}$ degree '0'.

Because:

Since x, y are differentiable functions of two independent variables u and v

$$\therefore dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$$
$$dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \quad \text{—— (1)}$$

Since Z is differentiable function of x and y and x, y are differentiable functions of u and v

$\therefore$ Z is a differentiable fn of u & v

$$\therefore dZ = \frac{\partial Z}{\partial u} du + \frac{\partial Z}{\partial v} dv \quad \text{—— (2)}$$

Also $dZ = \frac{\partial Z}{\partial x} dx + \frac{\partial Z}{\partial y} dy$

$$= \frac{\partial Z}{\partial x}\left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv\right) + \frac{\partial Z}{\partial y}\left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv\right) \quad (\text{from (1)})$$

$$= \left(\frac{\partial Z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial Z}{\partial y} \frac{\partial y}{\partial u}\right) du + \left(\frac{\partial Z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial Z}{\partial y} \frac{\partial y}{\partial v}\right) dv \quad \text{—— (3)}$$

Hence from (2) & (3),

$$\frac{\partial Z}{\partial u} du + \frac{\partial Z}{\partial v} dv = \left(\frac{\partial Z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial Z}{\partial y} \frac{\partial y}{\partial u}\right) du + \left(\frac{\partial Z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial Z}{\partial y} \frac{\partial y}{\partial v}\right) dv$$

$$\Rightarrow \boxed{\frac{\partial Z}{\partial u} = \frac{\partial Z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial Z}{\partial y} \frac{\partial y}{\partial u}}$$

$$\boxed{\frac{\partial Z}{\partial v} = \frac{\partial Z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial Z}{\partial y} \frac{\partial y}{\partial v}}$$

Problems

→ If $Z = e^{xy^2}$, $x = t\cos t$, $y = t\sin t$. Compute $\frac{dZ}{dt}$ at $t = \pi/2$.

Sol
$$\frac{dZ}{dt} = \frac{\partial Z}{\partial x}\frac{dx}{dt} + \frac{\partial Z}{\partial y}\cdot\frac{dy}{dt}$$
$$= (y^2 e^{xy^2})(\cos t - t\sin t) + (2xy\, e^{xy^2})(\sin t + t\cos t)$$

At $t = \frac{\pi}{2}$, $x = 0$, $y = \pi/2$

$$\therefore \left[\frac{dZ}{dt}\right]_{t=\pi/2} = \frac{\pi^2}{4}\left(-\pi/2\right) = \frac{-\pi^3}{8}$$

→ If $Z = x^3 - xy + y^3$, $x = r\cos\theta$, $y = r\sin\theta$.

find $\frac{\partial Z}{\partial r}$, $\frac{\partial Z}{\partial \theta}$.

Sol
$$\frac{\partial Z}{\partial r} = \frac{\partial Z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial Z}{\partial y}\frac{\partial y}{\partial r} = (3x^2 - y)\cos\theta + (3y^2 - x)\sin\theta$$

$$\frac{\partial Z}{\partial \theta} = \frac{\partial Z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial Z}{\partial y}\frac{\partial y}{\partial \theta}$$
$$= -(3x^2 - y)(-r\sin\theta) + (3y^2 - x)\, r\cos\theta$$

→ Show that $Z = f(x^2y)$, where f is differentiable, satisfies $x\left(\frac{\partial Z}{\partial x}\right) = 2y\left(\frac{\partial Z}{\partial y}\right)$

Sol Let $x^2y = u$ then $Z = f(u)$

$$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial u}\frac{\partial u}{\partial x} = f'(u)\cdot 2xy$$
$$\frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial u}\cdot\frac{\partial u}{\partial y} = f'(u)\cdot x^2$$
$$\therefore x\frac{\partial Z}{\partial x} = f'(u)\, 2x^2y = 2y\frac{\partial Z}{\partial y}$$

Aliter:- $dZ = f'(u)du = f'(x^2y)(2xy\,dx + x^2dy)$ — (1)

Also $dZ = \frac{\partial Z}{\partial x}dx + \frac{\partial Z}{\partial y}dy$ — (2)

from (1) & (2), we have $\frac{\partial Z}{\partial x} = 2xyf'(x^2y)$, $\frac{\partial Z}{\partial y} = x^2f'(x^2y)$. The result now follows as above

Euler Theorem on Homogenous functions:-

If $z = f(x, y)$ is homogenous function of x and y of degree 'n', then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz.$$

Cor:- If $z = f(x, y)$ is a homogenous function of x, y of degree 'n'

then $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z$

Problems:-

If $u = \cot^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{4}\sin 2u = 0.$$

Sol Let $u = \cot^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$.

then $\cot u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$ $(= z)$ say ——— (1)

clearly z is a homo. function of x and y of degree $\frac{1}{2}$

By eulers theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{2}(z) \text{ ——— (2)}$$

from (1), $\frac{\partial z}{\partial x} = -\csc^2 u \frac{\partial u}{\partial x}$.

$\frac{\partial z}{\partial y} = -\csc^2 u \frac{\partial u}{\partial y}$

$\therefore$ (2) $\equiv x\left(-\csc^2 u \frac{\partial u}{\partial x}\right) + y\left(-\csc^2 u\right)\frac{\partial u}{\partial y} = \frac{1}{2}($

$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\sin^2 u\left(\frac{1}{2}\frac{\cos u}{\sin u}\right)$

$= -\frac{1}{4}\sin 2u.$

If $u = \tan^{-1} \frac{x^3+y^3}{x-y}$, $x \neq y$ show that

(i) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$

(ii) $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = (1 - 4\sin^2 u) \sin 2u$.

Sol^n:
(i) Here $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ is not a homogeneous function.

However, we write

$$\tan u = \frac{x^3+y^3}{x-y} \; (= z) \text{ say} \quad \text{——— (1)}$$

$$\Rightarrow z = x^2 \left[\frac{1 + (y/x)^3}{1 - (y/x)} \right]$$

$\therefore$ z is a homogeneous function of x, y of degree 2.

$$\therefore x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2z \quad \text{——— (2)}$$

But from (1) $\frac{\partial z}{\partial x} = \sec^2 u \frac{\partial u}{\partial x}$ & $\frac{\partial z}{\partial y} = \sec^2 u \frac{\partial u}{\partial y}$ ←—— (3)

$\therefore$ (2) $\equiv$

$$x \left[\sec^2 u \frac{\partial u}{\partial x} \right] + y \sec^2 u \frac{\partial u}{\partial y} = 2z = 2\tan u$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \frac{\sin u}{\cos u} \cdot \cos^2 u$$

$$= \sin 2u. \quad \text{——— (4)}$$

(ii) From (3)

$$\frac{\partial^2 z}{\partial x^2} = \sec^2 u \frac{\partial^2 u}{\partial x^2} + 2\sec^2 u \tan u \left(\frac{\partial u}{\partial x}\right)^2$$

$$\frac{\partial^2 z}{\partial y^2} = \sec^2 u \frac{\partial^2 u}{\partial y^2} + 2\sec^2 u \tan u \left(\frac{\partial u}{\partial y}\right)^2$$

and $\frac{\partial^2 z}{\partial x \partial y} = \sec^2 u \frac{\partial^2 u}{\partial x \partial y} + 2\sec^2 u \tan u \frac{\partial u}{\partial x} \frac{\partial u}{\partial y}$.

By corollary of Euler's theorem, we have

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=2(2-1)z.$$

$$\Rightarrow \sec^2 u\left(x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}\right)$$

$$+2\sec^2 u\tan u\left[x^2\left(\frac{\partial u}{\partial x}\right)^2+2xy\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+y^2\left(\frac{\partial u}{\partial y}\right)^2\right]=2\tan u$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+2\tan u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)^2=2\sin u\cos u$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\sin 2u-2\tan u\sin^2 2u$$

$$=(1-2\tan u\sin 2u)\sin 2u$$

$$=(1-4\sin^2 u)\sin 2u.$$

→ If $z=(x+y)\,\phi(y/x)$, where ϕ is any arbitrary function prove that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$

Sol^n: $\frac{\partial z}{\partial x}=\phi(y/x)+(x+y)\,\phi'(y/x)\cdot\left(-\frac{y}{x^2}\right)$

$$\Rightarrow x\frac{\partial z}{\partial x}=x\,\phi(y/x)-\frac{y}{x}(x+y)\,\phi'(y/x) \quad \text{(1)}$$

Also $\frac{\partial z}{\partial y}=\phi(y/x)+(x+y)\,\phi'(y/x)\left(\frac{1}{x}\right)$

$$\Rightarrow y\frac{\partial z}{\partial y}=y\,\phi(y/x)+\frac{y}{x}(x+y)\,\phi'(y/x) \quad \text{(2)}$$

adding (1) & (2)

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=(x+y)\,\phi(y/x)$$

$$=z.$$

$$\therefore x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z.$$

→ If $u = z e^{ax+by}$, where z is a homogeneous function of x and y of degree n, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = (ax+by+n)u.$$

→ If $z = x^m f(y/x) + x^n g(y/x)$. prove that

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} + mnz = (m+n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$$

→ If $z = \log\left(\frac{x^4+y^4}{x+y}\right)$, then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3$

→ If $u = \sec^{-1}\left(\frac{x^3+y^3}{x+y}\right)$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u$

→ If $u = f(x+2y) + g(x-2y)$. Show that

$$4\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$$

2007 (15M) → If $w = \phi(x+at) + \psi(x-at)$. show that

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$$

→ If $z = f\left[\frac{ny-mz}{nx-lz}\right]$. prove that

$$(nx-lz)\frac{\partial z}{\partial x} + (ny-mz)\frac{\partial z}{\partial y} = 0$$

2006 (15M) If $z = x f(y/x) + g(y/x)$

Show that $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0.$

→ If $z = x^m f(y/x) + x^n g(x/y)$,

prove that $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} + mnz$

$$= (m+n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$$

Sol^n: Let $u = x^m f(y/x)$ and $v = x^n g(x/y)$

Then $z = u+v$ ——— (1)

Now $u = x^m f(y/x)$ is a homogeneous function in x and y of degree m.

Therefore by Euler's theorem, we have

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = m(m-1)u \quad \text{——— (2)}$$

Also, $v = x^n g(x/y)$ is a homogeneous function in x and y of degree n.

So we have

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} + y^2\frac{\partial^2 v}{\partial y^2} = n(n-1)v \quad \text{——— (3)}$$

Adding (2) & (3), we have

$$x^2\frac{\partial^2}{\partial x^2}(u+v) + 2xy\frac{\partial^2}{\partial x \partial y}(u+v) + \frac{\partial^2}{\partial y^2}(u+v)$$

$$= m(m-1)u + n(n-1)v$$

$$\Rightarrow x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = m(m-1)u + n(n-1)v \quad \text{——— (4)}$$

Now let $m(m-1)u + n(n-1)v$

$$= (m^2u + n^2v) - (mu + nv)$$

$$= m^2u + n^2v - mnu + mnu - mnv + mnv - (mu+nv)$$

$$= mu(m+n) + nv(m+n) - mn(u+v) - (mu+nv)$$

$$= (mu+nv)(m+n) - mn(u+v) - (mu+nv)$$

$$= (mu+nv)(m+n-1) - mn(u+v).$$

$$= -mnz + (mu+nv)(m+n-1) \quad \text{—(5)}$$

(∵ from (1) $z = u+v$)

Again from Euler's theorem we have for u & v, which are homogeneous functions in x & y of degree m and n respectively,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = mu \quad \text{and}$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv.$$

Adding these we get,

$$x\frac{\partial}{\partial x}(u+v) + y\frac{\partial}{\partial y}(u+v) = mu+nv.$$

$$\Rightarrow \boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = mu+nv} \quad (\because z = u+v)$$

∴ from (3)

we have

$$m(m-1)u + n(n-1)v = (m+n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) - mnz \quad \text{—(6)}$$

& ∴ from (4) we have

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (m+n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) - mnz$$

$$\Rightarrow x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + mnz = (m+n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right)$$

→ If $z = f\left(\frac{ny - mz}{nx - lz}\right)$

prove that $(nx - lz)\frac{\partial z}{\partial x} + (ny - mz)\frac{\partial z}{\partial y} = 0$

Sol[n] Given $z = f\left(\frac{ny - mz}{nx - lz}\right)$.

$$\frac{\partial z}{\partial x} = f'\left(\frac{ny - mz}{nx - lz}\right) \frac{\partial}{\partial x}\left(\frac{ny - mz}{nx - lz}\right)$$

$$= f'\left(\frac{ny - mz}{nx - lz}\right) \cdot (ny - mz) \cdot \frac{-n}{(nx - lz)^2} \quad \text{--- (1)}$$

$$\frac{\partial z}{\partial y} = f'\left(\frac{ny - mz}{nx - lz}\right) \frac{\partial}{\partial y}\left(\frac{ny - mz}{nx - lz}\right)$$

$$= f'\left(\frac{ny - mz}{nx - lz}\right) \frac{n}{nx - lz} \quad \text{--- (2)}$$

multiplying (1) by $(nx - lz)$ and (2) by $(ny - mz)$

and adding these, we get

$$(nx - lz)\frac{\partial z}{\partial x} + (ny - mz)\frac{\partial z}{\partial y} = (nx - lz) f'\left(\frac{ny - mz}{nx - lz}\right) \frac{-n(ny - mz)}{(nx - lz)^2}$$

$$+ (ny - mz) f'\left(\frac{ny - mz}{nx - lz}\right) \frac{n}{nx - lz}$$

$$= f'\left(\frac{ny - mz}{nx - lz}\right)\left[\frac{-n(ny - mz)}{nx - lz} + \frac{n(ny - mz)}{nx - lz}\right]$$

$$= f'\left(\frac{ny - mz}{nx - lz}\right) \cdot 0$$

$$= 0$$

$$\therefore (nx - lz)\frac{\partial z}{\partial x} + (ny - mz)\frac{\partial z}{\partial y} = 0$$

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Taylor's theorem for function of two variables

Monomial:

→ Defn: Let x & y denote two variables. Then an expression of the form $a_{jk}x^j y^k$, where j and k are non-negative integers and $a_{jk} \in \mathbb{R}$, is called a monomial. The integer $j+k$ is called the degree of the monomial.

For example:

x^2y^3 is a monomial of degree 5.

x^4 is a monomial of degree 4

y^7 is a monomial of degree 7.

→ A polynomial in two variables in x & y with coefficients in $\mathbb{R}$ is in an expression of the type

$$P(x,y) = a_{00} + (a_{10}x + a_{01}y) + (a_{20}x^2 + a_{11}xy + a_{02}y^2) + \cdots$$
$$+ (a_{i0}x^i + a_{(i-1)1}x^{i-1}y + \cdots + a_{0i}y^i) + \cdots + (a_{n0}x^n + \cdots + a_{0n}y^n)$$

where a_{ij}'s are real numbers.

In the first bracket, each term is a monomial of degree 1.

In the second, each is a monomial of degree 2,

and so on.

For example:

$P(x,y) = 1 + 2xy + x^2y$ is a polynomial in two variables.

This polynomial is a sum of three monomials, having degree 0, 2 and 3 respectively.

The number 3, which is the maximum of these numbers is called the degree of this polynomial.

→ The highest degree of the monomials present in a polynomial $p(x,y)$ is called the degree of $p(x,y)$.

→ n^{th} Taylor polynomial of a function of two variables

Defn: Let $f(x,y)$ be a real-valued function of two variables. Assume that it has continuous partial derivatives of all types of orders less than or equal to 'n' in some nbd of a point (x_0, y_0).

Then
$$T_n(x,y) = \sum_{i,j=0}^{i+j\le n} \frac{1}{i!\,j!}\left[\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(x_0, y_0)\right](x-x_0)^i (y-y_0)^j$$
is called the n^{th} Taylor polynomial of f at (x_0,y_0)

In particular, if $f(x,y)$ is a polynomial of degree 'n', then all partial derivatives of order m for $m>n$ will be zero.

$\therefore\ T_m(x,y) = T_n(x,y)$ for all $m \geq n$.

Further, as in the case of one variable, we can see that $T_n(x,y)$ at $(0,0)$ is equal to $f(x,y)$.

From the defn, we can see that
$$T_{n+1}(x,y) = T_n(x,y) + \sum_{\substack{i,j=0\\ i+j=n+1}} \frac{1}{i!\,j!}\left[\frac{\partial^{i+j} f(x_0,y_0)}{\partial x^i \partial y^j}\right](x-x_0)^i (y-y_0)^j$$

→ Find the Taylor polynomials of the function $p(x,y) = 1+2xy+x^2y$ at $(1,1)$.

Soln: Given $p(x,y) = 1+2xy+x^2y$

$p(1,1) = 4$.

By defn
$$T_n(x,y) = \sum_{i,j=0}^{i+j\le n} \frac{1}{i!\,j!}\left[\frac{\partial^{i+j} p(x_0,y_0)}{\partial x^i \partial y^j}\right](x-x_0)^i (y-y_0)^j \quad \text{--- (1)}$$

put $n=0$ in ①

$$T_0(x,y) = \frac{1}{0!\,0!}\left[\frac{\partial^{0+0} f(x_0,y_0)}{\partial x^0 \partial y^0}\right](x-x_0)^0 (y-y_0)^0$$

$$= f(x_0,y_0)$$

$$= f(1,1) = 4.$$

$\therefore T_0(x,y) = 4$

put $n=1$ in ①

$$T_1(x,y) = \sum_{i,j=0}^{i+j\le 1} \frac{1}{i!\,j!}\left[\frac{\partial^{i+j} f(x_0,y_0)}{\partial x^i \partial y^j}\right](x-x_0)^i (y-y_0)^j$$

$$= f(x_0,y_0) + \frac{1}{1!\,0!}\frac{\partial f(x_0,y_0)}{\partial x}(x-x_0) + \frac{1}{0!\,1!}\frac{\partial f}{\partial y}(x_0,y_0)(y-$$

$$= T_0(x,y) + \frac{\partial f}{\partial x}(1,1)(x-1) + \frac{\partial f}{\partial y}(1,1)(y-1) \quad — ②$$

$(\because f(1,1) = T_0(x,y))$

NOW

$$\frac{\partial f}{\partial x} = 2y + 2xy \Rightarrow \left(\frac{\partial f}{\partial x}\right)_{(1,1)} = 4$$

$$\frac{\partial f}{\partial y} = 2x + x^2 \Rightarrow \left(\frac{\partial f}{\partial y}\right)_{(1,1)} = 3.$$

$\therefore$ ② $\equiv$

$$T_1(x,y) = 4 + 4(x-1) + 3(y-1)$$

put $n=2$ in ① $\quad \& \ (x_0,y_0) = (1,1)$

$$T_2(x,y) = \sum_{i,j=0}^{i+j\le 2}\left[\frac{1}{i!\,j!}\ \frac{\partial^{i+j} f(1,1)}{\partial x^i \partial y^j}\right](x-1)^i (y-1)^j$$

$$= T_1(x,y) + \frac{(x-1)^2}{2!}\frac{\partial^2 f(1,1)}{\partial x^2} + \frac{(x-1)(y-1)}{1!\,1!}\frac{\partial^2 f}{\partial x \partial y}(1,1)$$

$$+ \frac{(y-1)^2}{2!}\frac{\partial^2 f}{\partial y^2}(1,1) \quad — ③$$

NOW $\frac{\partial^2 f}{\partial x^2} = 2y \Rightarrow \left(\frac{\partial^2 f}{\partial x^2}\right)_{(1,1)} = 2$

$$\frac{\partial^2 f}{\partial x \partial y} = 2 + 2x \Rightarrow \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(1,1)} = 4$$

$\frac{\partial^2 p}{\partial y^2} = 0 \Rightarrow \left(\frac{\partial^2 p}{\partial y^2}\right)_{(1,1)} = 0$

substituting these values in (8);

we get

$T_2(x,y) = 4 + 4(x-1) + 3(y-1) + (x-1)^2 + 4(x-1)(y-1)$

Since $\frac{\partial^3 p}{\partial x^3} = 0$, $\frac{\partial^3 p}{\partial x^2 \partial y} = 2$, $\frac{\partial^3 p}{\partial x \partial y^2} = 0$ and $\frac{\partial^3 p}{\partial y^3} = 0$

we get

$T_3(x,y) = T_2(x,y) + (x-1)^2(y-1)$

and $T_r(x,y) = T_3(x,y)$ for all $r \geq 3$.

→ Find the Taylor polynomial $T_3(x,y)$ for the function $\sin(x+y)$ at $(0,0)$.

Sol^n: Let $f(x,y) = \sin(x+y)$.

Clearly f has continuous partial derivatives of all orders.

Also $f(0,0) = 0$.

Now, $\frac{\partial f}{\partial x} = \cos(x+y) = \frac{\partial f}{\partial y}$

$\Rightarrow \left(\frac{\partial f}{\partial x}\right)_{(0,0)} = \left(\frac{\partial f}{\partial y}\right)_{(0,0)} = 1$.

$\frac{\partial^2 f}{\partial x^2} = -\sin(x+y) = \frac{\partial^2 f}{\partial y^2}$

$\Rightarrow \left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,0)} = \left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,0)} = 0$

and $\frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y)$

$\Rightarrow \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(0,0)} = 0$.

$\frac{\partial^3 f}{\partial x^3} = \frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial^3 f}{\partial y^3} = -\cos(x+y)\Big|_{(0,0)} = -1$.

The third Taylor polyonial of $\sin(x+y)$ at $(0,0)$ is

$$T_3(x,y) = \sum_{i,j=0}^{i+j\le 3} \frac{1}{i!\,j!}\left[\frac{\partial^{i+j} f(0,0)}{\partial x^i \partial y^j}\right] x^i y^j$$

$$= \frac{1}{0!\,0!} f(0,0) + \frac{x}{1!\,0!}\frac{\partial f}{\partial x}(0,0) + \frac{y}{0!\,1!}\frac{\partial f}{\partial y}(0,0) + \frac{x^2}{2!\,0!}\frac{\partial^2 f}{\partial x^2}(0,0) + \frac{xy}{1!\,1!}\frac{\partial^2 f}{\partial x \partial y}(0,0) + \frac{y^2}{0!\,2!}\frac{\partial^2 f}{\partial y^2}(0,0)$$

$$+ \frac{1}{3!\,0!}\left(\frac{\partial^3 f(0,0)}{\partial x^3}\right)x^3 + \frac{1}{2!\,1!}\left(\frac{\partial^3 f(0,0)}{\partial x^2 \partial y}\right)x^2y + \frac{1}{1!\,2!}\left(\frac{\partial^3 f(0,0)}{\partial x \partial y^2}\right)xy^2 + \frac{1}{3!}\frac{\partial^3 f(0,0)}{\partial y^3}y^3$$

$$= 0 + \frac{x}{1!} + \frac{y}{1!} - 0 - \frac{1}{3!}x^3 - \frac{1}{2!\,1!}x^2y - \frac{1}{1!\,2!}xy^2 - \frac{1}{3!}y^3$$

$$\Rightarrow T_3(x,y) = (x+y) - \frac{1}{3!}(x^3+3x^2y+3xy^2+y^3)$$

$$= (x+y) - \frac{(x+y)^3}{3!}$$

H.W. → Find the second Taylor polynomial of e^{x+y} at $(0,0)$

H.W. → Find the Taylor polynomials of $f(x,y) = 2+x^2+y^3$ at $(1,$

→ Now let us consider a function $f(x,y)$ of two variables. ~~Assume that~~ f has continuous partial derivatives of all ~~orders less than or equal to~~ n, for some integer 'n'; in a nbd of a point (x_0,y_0).

Then the n^{th} Taylor polynomial

$$T_n(x,y) = \sum_{i,j=0}^{i+j\le n} \frac{1}{i!\,j!}\left(\frac{\partial^{i+j} f}{\partial x^i \partial y^j}\right)_{(x_0,y_0)} (x-x_0)^i (y-y_0)^j$$

has the same value as $f(x,y)$ at (x_0,y_0), and the same partial derivatives ~~of all orders $\le n$ as f at (x_0,y_0)~~

As in the case of one variable, we would naturally like to know whether we can approximate 'f' by the corresponding Taylor polynomials.

Put differently, we would like to have some information about the function

$$R_{n+1}(x,y) = f(x,y) - T_n(x,y).$$

An analogue of Taylor's theorem which we state now, provides us some information about the function $R_{n+1}(x,y)$.

Taylor's theorem:

Let 'f' be a real-valued function of two variables x and y with continuous partial derivatives of orders $\leq n+1$ in some nbd of $S(\bar{x}, r)$ of $\bar{x} = (x_0, y_0)$. Then for a given $(x,y) \neq (x_0,y_0)$ in $S(\bar{x}, r)$, there exists a point (c_1, c_2) on the line segment joining (x_0,y_0) and (x,y) such that —— (1)

$$f(x,y) = T_n(x,y) + R_{n+1}(x,y)$$

where,
$$T_n(x,y) = \sum_{i,j=0}^{i+j\leq n} \frac{1}{i!\,j!}\left(\frac{\partial^{i+j} f}{\partial x^i \partial y^j}\right)_{(x_0,y_0)} (x-x_0)^i (y-y_0)^j$$

and
$$R_{n+1}(x,y) = \sum_{i+j=n+1}\left(\frac{1}{i!\,j!}\frac{\partial^{i+j} f}{\partial x^i \partial y^j}\right)_{(c_1,c_2)} (x-x_0)^i (y-y_0)^j$$

i.e.,
$$R_{n+1}(x,y) = \frac{1}{(n+1)!}\left(\frac{\partial^{n+1}f}{\partial x^{n+1}}\right)_{(c_1,c_2)} (x-x_0)^{n+1} +$$
$$\frac{1}{n!\,1!}\left(\frac{\partial^{n+1} f}{\partial x^n \partial y}\right)_{(c_1,c_2)} (x-x_0)^n (y-y_0) +$$
$$\frac{1}{(n-1)!\,2!}\left(\frac{\partial^{n+1} f}{\partial x^{n-1}\partial y^2}\right)_{(c_1,c_2)} (x-x_0)^{n-1}(y-y_0)^2 + \cdots$$
$$\cdots + \frac{1}{(n+1)!}\left(\frac{\partial^{n+1} f}{\partial y^{n+1}}\right)_{(c_1,c_2)} (y-y_0)^{n+1}.$$

i.e, $R_{n+1}(x,y)$ involves all the $(n+1)^{th}$ order partial derivatives of f evaluated at the point (c_1, c_2).

The RHS of (1) is called the n^{th} Taylor expansion of f at (x_0, y_0).

Now we consider only the second Taylor expansion of functions.

If we look at the expression for $R_3(x,y)$, we will see that it contains powers of $(x-x_0)$ and $(y-y_0)$. Now if we take the point (x,y) close enough to (x_0, y_0), then $(x-x_0)$ and $(y-y_0)$ will be very small.

Therefore, we can get a good enough approximation of $f(x,y)$ by a second degree polynomial. Of course, $f(x,y)$ can be approximated as closely as we like by a polynomial by choosing n sufficiently large.

We write the expression for $T_2(x,y)$ and the second Taylor expansion of $f(x,y)$ at (x_0, y_0) explicitly:

$$f(x,y) = f(x_0,y_0) + \left[\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)\right]$$
$$+ \frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2}(x_0,y_0)(x-x_0)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)(x-x_0)(y-y_0)\right.$$
$$\left. + \frac{\partial^2 f}{\partial y^2}(x_0,y_0)(y-y_0)^2\right] + R_3(x,y).$$

$$= f(x_0,y_0) + \left[(x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right] f(x_0,y_0)$$
$$+ \frac{1}{2!}\left[(x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right]^2 f(x_0,y_0) + R_3(x,y)$$

→ Find the second Taylor expansion of the function $f(x,y) = \log(1+x+2y)$. for points close to $(2,1)$.

Sol: Given $f(x,y) = \log(1+x+2y)$

$$f(2,1) = \log 5.$$

$$\frac{\partial f}{\partial x} = \frac{1}{1+x+2y} \Rightarrow \left(\frac{\partial f}{\partial x}\right)_{(2,1)} = \frac{1}{5}$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+x+2y}\cdot(2) \Rightarrow \left(\frac{\partial f}{\partial x}\right)_{(2,1)} = \frac{2}{5}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{-1}{(1+x+2y)^2} \Rightarrow \left(\frac{\partial^2 f}{\partial x^2}\right)_{(2,1)} = \frac{-1}{25}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-4}{(1+x+2y)^2} \Rightarrow \left(\frac{\partial^2 f}{\partial y^2}\right)_{(2,1)} = -\frac{4}{25}.$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{-2}{(1+x+2y)^2} \Rightarrow \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(2,1)} = \frac{-2}{25}$$

∴ The second Taylor expansion is given by

$$f(x,y) = \log 5 + \left[\frac{1}{5}(x-x_0) + \frac{2}{5}(y-y_0)\right]$$
$$+ \frac{1}{2}\left[\left(\frac{-1}{25}\right)(x-x_0)^2 + (2)\left(\frac{-2}{5}\right)(x-x_0)(y-y_0)\right.$$
$$\left. + \left(\frac{-4}{25}\right)(y-y_0)^2\right]$$

→ Find the second Taylor expansion for the function $f(x,y) = xy^2 + \cos xy$ about $(1, \pi/2)$.

→ Find an approximation to the function $f(x,y) = e^x \sin y$ by a second degree polynomial near $(0,0)$.

EXTREME VALUES

Maxima and Minima

Defn: A function $f: D \to \mathbb{R}$, $D \subset \mathbb{R}^n$, is said to have maximum at 'a' if there exists a nbd of 'a' for every point x of which $f(x) < f(a)$.

i.e, if $f(x,y)$ be a real valued function of two variables, we say that the function f has a maximum at (a,b) if $f(x,y) < f(a,b)$ for every $(x,y) \in N_\delta(a,b)$ for some δ;

Similarly, $f: D \to \mathbb{R}$, $D \subset \mathbb{R}^n$ is said to have a minimum at 'a' if there exists a nbd of 'a' at every point of which $f(x) > f(a)$.

i.e, if $f(x,y)$ be a real valued function of two variables, we say that the function f has a minimum at (a,b) if $f(x,y) > f(a,b)$ for every $(x,y) \in N_\delta(a,b)$ for some $\delta > 0$.

(OR)

→ Let (a,b) be a point of the domain of definition of function f. Then $f(a,b)$ is an extreme value of f, if for every point (x,y), (other than (a,b)) of some nbd of (a,b), the difference

$$f(x,y) - f(a,b) \quad \text{——— (1)}$$

keeps the same sign.

The extreme value $f(a,b)$ is called a max or min. value according as the sign of (1) is $-$ve or $+$ve.

→ A function f is said to have an extreme value at (a,b), if $f(a,b)$ is either a maximum or minimum

value of the function.

→ A necessary condition for $f(a,y)$ to have an extreme value at (a,b) is that $f_x(a,b)=0$, $f_y(a,b)=0$, provided these partial derivatives exist.

(or)

Let f be a function of two variables. Suppose f has an extremum at some point (a,b) and the partial derivatives of f exist at that point. Then

$$f_x(a,b) = 0 = f_y(a,b).$$

→ To check whether a given function has an extremum at some point or not, we can use above theorem. All we have to do is to see whether its partial derivatives vanish at that point (if they exist).

(1) ex: Check whether the function given by $f(x,y) = x^2 - 2x + \frac{y^2}{4}$ has maximum or minimum values.

sol: The given function $f(x,y) = x^2-2x+\frac{y^2}{4}$ is differentiable everywhere.

First we have to find out the points (x,y) such that $f_x(x,y) = 0 = f_y(x,y)$.

Now $f_x(x,y) = 2x-2$.

$f_y(x,y) = \frac{y}{2}$.

$\therefore$ $f_x(x,y)$ and $f_y(x,y)$ will vanish only when $x=1$ and $y=0$.

$\therefore$ the point $(1,0)$ is the only possible point where f can have a max or min value.

Now, let us see whether $(1,0)$ is a max or min. point for f.

$$f(x,y) = x^2 - 2x + \frac{y^2}{4}$$
$$= x^2 - 2x + 1 + \frac{y^2}{4} - 1$$
$$= (x-1)^2 + \frac{y^2}{4} - 1$$

$f(1,0) = 1-2 = -1$

$f(x,y) - f(1,0) = x^2 - 2x + \frac{y^2}{4} + 1$

$= (x-1)^2 + \frac{y^2}{4} \geq 0$

$\therefore f(x,y) - f(1,0) \geq 0$

$\therefore f(x,y) \geq f(1,0)$

This shows that $f(x,y) \geq -1 = f(1,0)$. $\forall (x,y)$.

$\therefore$ The function f has minimum at $(1,0)$.

The minimum value is $f(1,0) = -1$.

and the function has no maximum value.

Note: If $f_x \neq 0$ or $f_y \neq 0$ at some point, then we can straightaway say that the function does not have an extremum at that point.

But if $f_x = 0 = f_y$ at some point, then this does not imply that the function has extremum at that point.

It is possible that all the first order partial derivatives of a function are zero at some point (a,b), but still, that point is not an extremum point for that function.

i.e., the converse of the above theorem is not true.

For example

Consider the function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$f(x,y) = 1 - x^2 + y^2.$$

Sol: We have $f_x = -2x$ & $f_y = 2y$

$\therefore f_x(0,0) = 0 = f_y(0,0)$.

Now, let us check whether the function f has an extremum at $(0,0)$.

We have $f(0,0) = 1$.

$f(a, 0) < 1$ and $f(0, b) > 1$ for all non-zero 'a' and 'b'

$\therefore$ In the nbd of $(0,0)$, we can find points of the type $(a,0)$ and $(0,b)$.

$\therefore$ there exists no nbd N of $(0,0)$ for which

$f(x,y) < f(0,0)$ or $f(x,y) > f(0,0)$.
$\forall (x,y) \in N$.

$\therefore$ $(0,0)$ is neither a maximum nor a minimum point for f, even though ~~both~~ the partial derivatives of f vanish at $(0,0)$

Ex-2: ~~If~~ $f(x,y) = 0$ if $x=0$ or $y=0$
$= 1$ elsewhere.

then both the partial derivatives exist (each equal to zero) at the origin, but $f(0,0)$ is not an extreme value.

Thus the conditions obtained in the above theorem are only necessary and not sufficient.

Some times it may happen that the partial derivatives of a function donot exist at a point, but still the function has an extremum at that point.

for example :

Consider the function given by

$f(x,y) = 1+\sqrt{x^2+y^2}$.

Sol^n: Since $f(0,0) = 1$

$f(x,y) = 1+\sqrt{x^2+y^2} > 1 = f(0,0)$

i.e., $f(x,y) > f(0,0)$ for every point (x,y) in the nbd of $(0,0)$.

It follows that f has a minimum at $(0,0)$.

Now, $f_x(0,0) = \lim_{h\to 0} \dfrac{f(0+h,0)-f(0,0)}{h}$

$= \lim_{h\to 0} \dfrac{1+\sqrt{h^2}-1}{h} = \lim_{h\to 0} \dfrac{\sqrt{|h|^2}}{h}$

$= \lim_{h\to 0} \dfrac{|h|}{h}$

which does not exist.

~~Similarly~~ $\therefore f_x$ doesnot exist at $(0,0)$

Similarly f_y doesnot exist at $(0,0)$

Ex2: The function $f(x,y) = |x|+|y|$ has an extreme value at $(0,0)$ even though the partial derivatives f_x and f_y donot exist at $(0,0)$.

Sol^n: Since $f(0,0) = 0 < |x|+|y| = f(x,y)$

i.e., $f(x,y) > f(0,0)$ for every point (x,y) in the nbd of $(0,0)$

It follows that f has a minimum at $(0,0)$

Now for the existence of partial derivatives of f at $(0,0)$:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h, 0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{|h| - 0}{h}$$

$$= \lim_{h\to 0} \frac{|h|}{h}$$

which doesnot exist.

$\therefore f_x$ doesnot exist at $(0,0)$.

Similarly f_y doesnot exist at $(0,0)$.

Defn: Let f be a function of two variables. A point (a,b) is said to be a stationary point of f if both the partial derivatives are zero at (a,b).

Sufficient condition for $f(x,y)$ to have an extreme value at (a,b):

Theorem: If $f(x,y)$ has an extreme value at (a,b) and second order partial derivatives of $f(x,y)$ are continuous at (a,b) such that $f_x(a,b)=0$ and $f_y(a,b)=0$ and $(f_{xx}f_{yy} - f_{xy}^2)(a,b) > 0$ then $f(a,b)$ is a max or min according as f_{xx} (or f_{yy}) is $-$ve or $+$ve at (a,b)

i.e, $\rightarrow f_{xx}f_{yy}(a,b) - f_{xy}^2(a,b) > 0$

and $f_{xx}(a,b) < 0$ or $f_{yy}(a,b) < 0$

then f has a maximum at (a,b)

$\rightarrow f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}^2(a,b) > 0$

and $f_{xx}(a,b) > 0$ or $f_{yy}(a,b) > 0$

then f has minimum at (a,b).

Note 1: Further investigation is necessary, if

$f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}^2(a,b) = 0$

Note 2: $f_{xx}(a,b)\, f_{yy}(a,b) - f_{xy}^2(a,b) < 0$, then f has neither max nor minimum

problems

(1) Find the maxima and minima of the function

$f(x,y) = x^3 + y^3 - 3x - 12y + 20$.

Solⁿ: $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

$f_x(x,y) = 3x^2 - 3$

$f_y(x,y) = 3y^2 - 12$.

Equating to zero these f_x and f_y

we get $3x^2 - 3 = 0$

$\Rightarrow x = \pm 1$

and $3y^2 - 12 = 0$

$y = \pm 2$

$\therefore$ The function f has four stationary points

$(1,2), (-1,2), (1,-2), (-1,-2)$.

Now $f_{xx}(x,y) = 6x$

$f_{xy}(x,y) = 0$

$f_{yy}(x,y) = 6y$

At $(1,2)$ $f_{xx} = 6 > 0$, $f_{yy} = 12 > 0$ and $f_{xy} = 0$

and $f_{xx} f_{yy} - f_{xy}^2 = 6 \times 12 - 0$

$= 72 > 0$

Hence $f(1,2)$ is the minimum point of $f(x,y)$.

i.e, $f(x,y)$ has minimum at $(1,2)$.

At $(-1,2)$ $f_{xx} = -6$, $f_{yy} = 12$ and $f_{xy} = 0$

$$f_{xx} f_{yy} - f_{xy}^2 = -72 < 0.$$

$\therefore$ The f^n $f(x,y)$ has neither max. nor min. at $(-1,2)$

At $(1,-2)$

$f_{xx} = 6$, $f_{xy} = 0$ and $f_{yy} = -12$

and $f_{xx} f_{yy} - f_{xy}^2 = -72 < 0$

$\therefore$ The function $f(x,y)$ has neither max nor min at $(1,-2)$.

At $(-1,-2)$

$f_{xx} = -6 < 0$, $f_{yy} = -12 < 0$ and $f_{xy} = 0$

and $f_{xx} \cdot f_{yy} - f_{xy}^2 = 72 > 0$

$\therefore$ $(x,y) = (-1,-2)$ is the max pt of $f(x,y)$.
i.e, $f(x,y)$ has maximum at $(-1,-2)$.

Note: Stationary points like $(-1,2)$, $(+1,-2)$ which are not extreme (neither max nor min) points are called the saddle points.

(1) Find the all the maxima and minima of $f(x,y) = x^3+y^3-63(x+y)+12xy$.

Sol: $f(x,y) = x^3+y^3-63(x+y)+12xy$

$f_x(x,y) = 3x^2-63+12y$

but $f_x(x,y) = 0$

$\Rightarrow 3x^2-63+12y = 0$

$\Rightarrow 3(x^2-21+4y) = 0$

$\Rightarrow x^2+4y = 21$ —— (1)

and $f_y(x,y) = 3y^2-63+12x$

and $f_y(x,y) = 0$

$\Rightarrow 3y^2 - 63 + 12x = 0$

$\Rightarrow y^2 + 4x = 21$ — ②

①−② ≡ $x^2 - y^2 + 4(y-x) = 0$

$\Rightarrow (x-y)(x+y) + 4(y-x) = 0$

$\Rightarrow (x-y)[x+y-4] = 0$

$\Rightarrow x-y=0$; $x+y=4$

$\Rightarrow \boxed{x=y}$; $\boxed{y=4-x}$

Now sub $x=y$ in ①

$x^2+4x-21=0$

$(x-3)(x+7)=0$

$\Rightarrow x=3,-7$

$\Rightarrow y=3,-7$

∴ $(3,3), (-7,-7)$ are stationary points.

Now sub $y=4-x$ in ①

$\Rightarrow x^2+4(4-x)=21$

$\Rightarrow x^2-4x+16=21$

$\Rightarrow x^2-4x-5=0$

$\Rightarrow (x-5)(x+1)=0$

$\Rightarrow x=5,-1$

sub $x=5$ in $y=4-x$; sub $x=-1$ in $y=4-x$

$y=-1$; $\Rightarrow y=5$

$(x,y)=(5,-1)$; $(x,y)=(-1,5)$.

∴ $(3,3), (-7,-7), (-1,5), (5,-1)$ are stationary points.

~~At (3,3)~~

Now $f_{xx}=6x$, $f_{yy}=6y$, $f_{xy}=12$

put $D = f_{xx}f_{yy} - f_{xy}^2 = 6x \cdot 6y - (12)^2$

$= 36xy - 144$

$\boxed{D = 36xy - 144}$

At $(3,3)$

$f_{xx} = 18 > 0$ and $D = 36 \times 9 - 144$
$= 324 - 144 = 180 > 0$

$\therefore$ the function $f(x,y)$ is minimum at $(3,3)$.

At $(-7,-7)$

$f_{xx} = -42 < 0$ and $D = 36(-7)(-7) - 144$
$= 36 \times 49 - 144 > 0$

$\therefore$ the function is maximum at $(-7,-7)$

At $(-1,5)$

$f_{xx} = -6 < 0$ and $D = 36(-1)(5) - 144 < 0$

$\therefore$ the function $f(x,y)$ has neither max nor min at $(-1,5)$

At $(1,-5)$

$f_{xx} = 6 > 0$ and $D = 36(1)(-5) - 144 < 0$

$\therefore$ the function has neither max nor min at $(-1,5)$.

$\therefore$ $(3,3)$, $(-7,-7)$ are called extreme points

(1) S.T $f(x,y) = 2x^4 - 3x^2y + y^2$ has neither a max nor a minimum at $(0,0)$ where $f_{xx}f_{yy} - f_{xy}^2 = 0$

Sol: $f(x,y) = 2x^4 - 3x^2y + y^2$

$f_x(x,y) = 8x^3 - 6xy$

$f_y(x,y) = -3x^2 + 2y$

also $f_x(0,0) = 0$ and $f_y(0,0) = 0$.

and $f_{xx} = 24x - 6y$ $f_{xy} = -6x$ and $f_{yy} = 2$

At $(0,0)$, $f_{xx}f_{yy} - f_{xy}^2 = 0(2) - 0 = 0$.

$\therefore f_{xx}f_{yy} - f_{xy}^2 = 0$.

So that it is a doubtful case, and so requires

Again $f(x,y) = 2x^4 - 3x^2y + y^2$

$= 2x^4 - 2x^2y - x^2y + y^2$

$= 2x^2(x^2-y) - y(x^2-y)$

$= (2x^2-y)(x^2-y)$

and $f(0,0) = 0$

Now let $f(x,y) - f(0,0) = (x^2-y)(2x^2-y)$

> 0 if $y < 0$ or $x^2 > y > 0$

< 0 for $y > x^2 > \frac{y}{2} \geq 0$. i.e., $y > x^2$ and $2x^2 > y$ $\Rightarrow x^2 > \frac{y}{2}$

i.e., $f(x,y) - f(0,0)$ doesnot keep the same sign near the origin.

Hence f has neither maximum nor minimum at the origin.

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① S.T the ~~function~~ $f(x,y) = (y-x)^4 + (x-2)^4$ has ~~a~~ minimum at $(2,2)$.

→ ② S.T $f(x,y) = y^2 + x^2y + x^4$ has a minimum at $(0,0)$.

Sol: $f(x,y) = y^2 + x^2y + x^4$.

$f_x(x,y) = 2xy + 4x^3$; $f_{xx} = 2y + 12x^2$

$f_y(x,y) = 2y + x^2$; $f_{yy} = 2$.

and $f_x(0,0) = 0$; $f_{xy} = 2x$.

$f_y(0,0) = 0$

At $(0,0)$ $f_{xx} = 0$, $f_{yy} = 2$ and $f_{xy} = 0$

$\therefore f_{xx}f_{yy} - f_{xy}^2 = 0$

So ~~that it is a~~ doubtful case and requires further investigation.

Now $f(x,y) = y^2 + x^2y + x^4$

$= \left(y + \frac{1}{2}x^2\right)^2 + \frac{3}{4}x^4$

and $f(x,y) - f(0,0) = \left(y+\frac{1}{2}x^2\right)^2 + \frac{3}{4}x^4 > 0 \;\forall (x,y)$ in the nbd of $(0,0)$

$\therefore f$ has a minimum at the origin.

Extreme Values of a function of n variables

A point $(a_1, a_2, \ldots a_n)$ is said to be an extreme point, and $f(a_1, a_2, \ldots a_n)$ an extreme ~~point~~ value of a funct f, if for every point $(x_1, x_2, \ldots x_n)$, other than $(a_1, a_2, \ldots a_n)$ of some nbd of $(a_1, a_2, \ldots a_n)$, the difference,

$f(x_1, x_2, \ldots x_n) - f(a_1, a_2, \ldots a_n)$ keeps the same sign.

The extreme value is a maximum or a minimum value according as the sign is $-ve$ or $+ve$.

→ The necessary conditions for $f(a_1, a_2, \ldots a_n)$ to be an extreme value of the function 'f' are that all the partial derivatives $f_{x_1}, f_{x_2}, f_{x_3} \ldots f_{x_n}$, in case they exist, vanish at $(a_1, a_2, \ldots a_n)$.

Since these are only necessary and not sufficient conditions therefore points which satisfy these conditions may not be extreme points. A point $(a_1, a_2, \ldots a_n)$ is called a stationary point if all the first order partial derivatives of the function vanish at that point. Thus the stationary points are determined by solving the following n equations simultaneously.

$$f_{x_1}(x_1, x_2, \ldots x_n) = 0$$
$$f_{x_2}(x_1, x_2, \ldots x_n) = 0$$
$$\cdots \cdots \cdots$$
$$f_{x_n}(x_1, x_2, \ldots x_n) = 0$$

For a function of n independent variables $x_1, x_2, \ldots x_n$ the condition can be given in a more compact form. i.e. if $(a_1, a_2, \ldots a_n)$ is a stationary point, then $df(a_1, a_2, \ldots a_n) = 0$.

$\because df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$ ...
$\Rightarrow df = f_x dx + f_y dy$ ...

i.e., the differential function vanishes at a stationary point. For, at the stationary point all the partial derivatives vanish and therefore

$$df(a_1, a_2, \ldots a_n) = f_{x_1}(a_1, a_2, \ldots a_n)\,dx_1 + f_{x_2}(a_1, a_2, \ldots a_n)\,dx_2 + \ldots + f_{x_n}(a_1, a_2, \ldots a_n)\,dx_n = 0$$

Conversely, when $df = 0$, the coefficients of the differentials $dx_1, dx_2, \ldots dx_n$ of independent variables, are seperately equal to zero.

Rule For a function $f(x, y, z)$ of three independent variables, sufficient conditions for (a, b, c) to be an extreme point are that

i) $df(a, b, c) = f_x\,dx + f_y\,dy + f_z\,dz = 0$, so that

$$f_x = f_y = f_z = 0$$

and

ii) $$d^2f(a, b, c) = f_{xx}(dx)^2 + f_{yy}(dy)^2 + f_{zz}(dz)^2 + 2f_{xy}\,dx\,dy + 2f_{yz}\,dy\,dz + 2f_{zx}\,dz\,dx.$$

keeps the same sign for arbitrary values of dx, dy, dz; the extreme point being a maxima or a minima according as the sign of d^2f is $-ve$ or $+ve$. The point will be neither a maxima nor a minima if d^2f doesnot keep the same sign; and requires further investigation, if d^2f keeps the same sign but vanishes at some points of a nbd of (a, b, c).

$df = f_x\,dx + f_y\,dy + f_z\,dz$.

$$d^2f = d(df) = d(f_x\,dx + f_y\,dy + f_z\,dz) = d(f_x\,dx) + d(f_y\,dy) + d(f_z\,dz)$$

$$= (f_{xx}\,dx + f_{yx}\,dy + f_{zx}\,dz)\,dx + f_x\,d^2x + (f_{xy}\,dx + f_{yy}\,dy + f_{zy}\,dz)\,dy + f_y\,d^2y + (f_{xz}\,dx + f_{yz}\,dy + f_{zz}\,dz)\,dz + f_z\,d^2z.$$

but $f_x = f_y = f_z = 0$ at (a, b, c)

$$\therefore d^2f(a, b, c) = f_{xx}(dx)^2 + f_{yy}(dy)^2 + f_{zz}(dz)^2 \ldots$$

The condition [that d^2f keeps the same] sign may be stated in terms of matrices, as follows.

Consider the matrix

$$\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

d^2f will always be +ve iff ~~the~~ three principal minors

$$\left| f_{xx} \right|, \quad \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}, \quad \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix}$$

are all +ve.

Then f has minimum at (a, b, c).

and d^2f will always negative iff their signs are alternatively negative and positive.; then f has maximum at (a, b, c)

(OR)

Let $f: D \to \mathbb{R}$, $D \subset \mathbb{R}^n$ be a function ~~possessing~~ continuous ~~partial derivatives~~ partial derivatives upto ~~the~~ second order in a nbd of a point 'a' at which all the first order partial derivatives vanish,

then

i) 'f' has minimum at 'a', if $D_1, D_2, \ldots D_n$ are all +ve

ii) 'f' has maximum at 'a' if $D_1, D_2, \ldots D_n$ are alternatively -ve and +ve.

where $D_1 = \left| d_{11} \right|$, $D_2 = \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix}$, $D_3 = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix}$, ...

and $d_{ij} = \dfrac{\partial^2 f}{\partial x_i \partial x_j}$

Examine the following function for extreme values:

$f(x,y,z) = 2x^2 + 3y^2 + 4z^2 - 3xy + 8z$.

Sol^n Given $f(x,y,z) = 2x^2 + 3y^2 + 4z^2 - 3xy + 8z$

$f_x = 4x - 3y$.

$f_y = 6y - 3x$

$f_z = 8z + 8$

Equating f_x, f_y, f_z to zero.

we get $4x - 3y = 0$

$x - 2y = 0$

$z + 1 = 0$

$\Rightarrow x = 0,\ y = 0$ and $z = -1$

$\therefore$ we get the only stationary point of f as $(0,0,-1)$.

Now at $(0,0,-1)$,

we have

$d_{11} = f_{xx} = 4$; $d_{12} = f_{xy} = -3$; $d_{13} = f_{xz} = 0$

$d_{21} = f_{yx} = -3$; $d_{22} = f_{yy} = 6$; $d_{23} = f_{yz} = 0$

$d_{31} = f_{zx} = 0$; $d_{32} = f_{zy} = 0$; $d_{33} = f_{zz} = 8$

Also $D_1 = d_{11} = 4 > 0$, $D_2 = \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix} = \begin{vmatrix} 4 & -3 \\ -3 & 6 \end{vmatrix}$

$= 24 - 9 = 15 > 0$

$D_3 = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix} = \begin{vmatrix} 4 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 8 \end{vmatrix}$.

$= 8(24 - 9) = 8 \times 15 = 120 > 0$.

Hence $f(x,y,z)$ has a minimum at $(0,0,-1)$

$f(x,y,z) = (ax+by+cz)\, e^{-\alpha^2x^2-\beta^2y^2-\gamma^2z^2}$ are

$\pm\sqrt{\frac{1}{2}(a^2\alpha^{-2}+b^2\beta^{-2}+c^2\gamma^{-2})/e}$ and $\sqrt{\frac{1}{2}(a^2\alpha^{-2}+b^2\beta^{-2}+c^2\gamma^{-2})/e}$

Sol: $f(x,y,z) = (ax+by+cz)\, e^{-\alpha^2x^2-\beta^2y^2-\gamma^2z^2}$

$$f_x = (ax+by+cz)\, e^{-\alpha^2x^2-\beta^2y^2-\gamma^2z^2}\cdot(-2\alpha^2x) + a\, e^{-\alpha^2x^2-\beta^2y^2-\gamma^2z^2}$$

$$= \left[a-(ax+by+cz)(+2x\alpha^2)\right] e^{-\alpha^2x^2-\beta^2y^2-\gamma^2z^2}$$

$$= \left[a-2\alpha^2x\sum ax\right] e^{-\sum\alpha^2x^2}$$

sly $f_y = \left[b-2\beta^2y\sum ax\right] e^{-\sum\alpha^2x^2}$

sly $f_z = \left[c-2\gamma^2z\sum ax\right] e^{-\sum\alpha^2x^2}$

Equating f_x, f_y, f_z to zero.

$f_x = \left(a-2\alpha^2x\sum ax\right) e^{-\sum\alpha^2x^2} = 0$

$\Rightarrow a-2\alpha^2x\sum ax = 0$, since $e^{-\sum\alpha^2x^2} \neq 0$

$f_y = 0$

$\Rightarrow b-2\beta^2y\sum ax = 0$ —(1)

$f_z = 0 \Rightarrow c-2\gamma^2z\sum ax = 0$

$\therefore x\sum ax = \frac{a}{2\alpha^2}$ —(2)

$y\sum ax = b/2\beta^2$ —(3)

$z\sum ax = c/2\gamma^2$ —(4)

Multiplying (2) by a, (3) by b, (4) by c and adding

$$(ax+by+cz)\sum ax = \frac{1}{2}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)$$

$$\sum ax \sum ax = \frac{1}{2}\sum a^2\alpha^{-2}$$

$$\Rightarrow \left(\sum ax\right)^2 = \frac{1}{2}\sum a^2\alpha^{-2}$$

$$\pm\sqrt{\tfrac{1}{2}\sum a^2\alpha^{-2}} = \pm k \text{ (say)}$$

Hence from (1) the stationary points are

$a - 2\alpha^2 x \sum ax = 0$

$a - 2\alpha^2 x (\pm k) = 0$ putting $\sum ax = \pm k$ in (1)

$\Rightarrow \pm 2\alpha^2 x k = a$

$\Rightarrow \quad x = \pm \frac{a}{2\alpha^2 k}$.

$|||^{ly}$ $y = \pm \frac{b}{2\beta^2 k}$, $z = \pm \frac{c}{2\gamma^2 k}$.

putting $\sum ax = -k$ in (1)

$x = -\frac{a}{2\alpha^2 k}$, $y = \frac{-b}{2\beta^2 k}$, $z = \frac{-c}{2\gamma^2 k}$

$\therefore$ the stationary points are

$\left(\frac{a}{2\alpha^2 k}, \frac{b}{2\beta^2 k}, \frac{c}{2\gamma^2 k}\right)$, $\left(\frac{-a}{2\alpha^2 k}, \frac{-b}{2\beta^2 k}, \frac{-c}{2\gamma^2 k}\right)$

Again, we have

$f_{xx} = [0 - 2\alpha^2 x a - 2\alpha^2 \sum ax] e^{-\sum \alpha^2 x^2} + [a - 2\alpha^2 x \sum ax] -2\alpha^2 x e^{-\sum \alpha^2 x^2}$

$= -2\alpha^2 x [a - 2\alpha^2 x \sum ax] e^{-\sum \alpha^2 x^2}$
$\quad - 2\alpha^2 [\sum ax + ax] e^{-\sum \alpha^2 x^2}$

$f_{xy} = [b - 2\beta^2 y \sum ax] e^{-\sum \alpha^2 x^2} (-2\alpha^2 x)$
$\quad + (-2\beta^2 y a) e^{-\sum \alpha^2 x^2}$

$= -2\alpha^2 x (b - 2\beta^2 y \sum ax) e^{-\sum \alpha^2 x^2} - 2\beta^2 y a e^{-\sum \alpha^2 x^2}$

and similar expressions for f_{yy}, f_{zz}, f_{yz}, f_{zx}.

At the stationary point $\left(\frac{a}{2\alpha^2 k}, \frac{b}{2\beta^2 k}, \frac{c}{2\gamma^2 k}\right)$

we have $\sum a^2 \alpha^2 = \frac{1}{2}$.

$$f_{xx} = 0 - 2\alpha^2\left[k + \frac{a^2}{2\alpha^2 k}\right] e^{-1/2}$$

$$= -\frac{1}{\sqrt{e}}\left[2\alpha^2 k + \frac{a^2}{k}\right] = -\left(\frac{2\alpha^2 k^2 + a^2}{k\sqrt{e}}\right)$$

$$f_{yy} = -\left(\frac{2\beta^2 k^2 + b^2}{k\sqrt{e}}\right), \quad f_{zz} = -\left(\frac{2\gamma^2 k^2 + c^2}{k\sqrt{e}}\right)$$

$$f_{xy} = 0 - \frac{ab}{k\sqrt{e}} = \frac{-ab}{k\sqrt{e}}, \quad f_{yz} = \frac{-bc}{k\sqrt{e}}; \quad f_{zx} = \frac{-ca}{k\sqrt{e}}.$$

$$\therefore d^2 f = -\frac{1}{k\sqrt{e}}\left[(2\alpha^2 k^2 + a^2)dx^2 + (2\beta^2 k^2 + b^2)dy^2 + (2\gamma^2 k^2 + c^2)dz^2\right] - \frac{2}{k\sqrt{e}}\left(ab\,dx\,dy + bc\,dy\,dz + ca\,dz\,dx\right)$$

Now $f_{xx} = -\left(\frac{2\alpha^2 k^2 + a^2}{k\sqrt{e}}\right) < 0$

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -\frac{(2\alpha^2 k^2 + a^2)}{k\sqrt{e}} & \frac{-ab}{k\sqrt{e}} \\ \frac{-ab}{k\sqrt{e}} & -\frac{(2\beta^2 k^2 + b^2)}{k\sqrt{e}} \end{vmatrix}$$

$$= \frac{1}{k^2 e}\begin{vmatrix} 2\alpha^2 k^2 + a^2 & ab \\ ab & 2\beta^2 k^2 + b^2 \end{vmatrix}$$

$$= \frac{1}{k^2 e}\left(4\alpha^2\beta^2 k^4 + 2\alpha^2 k^2 b^2 + 2\beta^2 k^2 a^2 + a^2 b^2 - a^2 b^2\right)$$

$$= \frac{2}{e}\left(2\alpha^2\beta^2 k^2 + \alpha^2 b^2 + \beta^2 a^2\right) > 0$$

$$\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = -\frac{1}{k^3 e^{3/2}}\begin{vmatrix} 2\alpha^2 k^2 + a^2 & ab & ac \\ ab & 2\beta^2 k^2 + b^2 & bc \\ ac & bc & 2\gamma^2 k^2 + c^2 \end{vmatrix}$$

$$= \frac{1}{k^3 e^{3/2}}\Big[(2\gamma^2k^2+c^2)[4\alpha^2\beta^2k^2+2\alpha^2k^2b^2+2\beta^2k^2a^2+a^2b^2]$$

$$-bc\left[(bc)(2\alpha^2k^2+a^2)-a^2bc\right]+ac\left[abc-(2\beta^2k^2+b^2)ac\right]\Big]$$

$$= -\frac{1}{k^3e^{3/2}}\left[(2\gamma^2k^2+c^2)2k^2(\alpha^2\beta^2+\alpha^2b^2+a^2\beta^2)-2b^2c^2\alpha^2k^2-2a^2c^2\beta^2k^2\right]$$

$$= \frac{-4k}{e^{3/2}}\left(2\alpha^2\beta^2\gamma^2k^2+\alpha^2b^2c^2+\alpha^2\beta^2c^2+a^2b^2\gamma^2\right)<0$$

Thus the three principal minors have alternatively −ve and +ve signs and so d^2f is always −ve.

$\therefore \left(\frac{a}{2\alpha^2k}, \frac{b}{2\beta^2k}, \frac{c}{2\gamma^2k}\right)$ is a point of maxima.

and the maximum value $= ke^{-1/2}$

$$= \sqrt{\frac{1}{2}\Sigma a^2\alpha^{-2}}\sqrt{\frac{1}{e}}$$

$$= \sqrt{\frac{1}{2}\Sigma a^2\alpha^{-2}/e}$$

At the point $\left(\frac{-a}{2\alpha^2k}, \frac{-b}{2\beta^2k}, \frac{-c}{2\gamma^2k}\right)$, it may be shown as above that $\Sigma\alpha^2x^2=\frac{1}{2}$ and

$$f_{xx} = \frac{2\alpha^2k^2+a^2}{k\sqrt{e}}, \quad f_{yy}=\frac{2\beta^2k^2+b^2}{k\sqrt{e}}, \quad f_{zz}=\frac{2\gamma^2k^2+c^2}{k\sqrt{e}}$$

$$f_{xy}=\frac{ab}{k\sqrt{e}}, \quad f_{yz}=\frac{bc}{k\sqrt{e}}, \quad f_{zx}=\frac{ca}{k\sqrt{e}}.$$

and the three principal minors are of +ve signs.

So that d^2f is +ve.

$\therefore \left(\frac{-a}{2\alpha^2k}, \frac{-b}{2\beta^2k}, \frac{-c}{2\gamma^2k}\right)$ is a point of minima

and the minimum value of the function $= -ke^{-1/2}$

$$= -\sqrt{\frac{1}{2}\Sigma a^2\alpha^{-2}/e}.$$

→ S.T $f(x,y,z) = (x+y+z)^3 - 3(x+y+z) - 24xyz + a^3$ has a minima at $(1,1,1)$ and a maxima at $(-1,-1,-1)$.

Sol: Given $f(x,y,z) = (x+y+z)^3 - 3(x+y+z) - 24xyz + a^3$.

$$f_x = 3(x+y+z)^2 - 24yz - 3$$
$$f_y = 3(x+y+z)^2 - 24zx - 3$$
$$f_z = 3(x+y+z)^2 - 24xy - 3$$

∴ the stationary points are given by

$$(x+y+z)^2 - 8yz - 1 = 0 \quad — (1)$$
$$(x+y+z)^2 - 8zx - 1 = 0 \quad — (2)$$
$$(x+y+z)^2 - 8xy - 1 = 0 \quad — (3)$$

(2) − (1) $\quad z(x-y) = 0 \Rightarrow z = 0$ or $y = x$

(3) − (2) $\quad x(y-z) = 0 \Rightarrow x = 0$ or $y = z$

(1) − (3) $\quad y(z-x) = 0 \Rightarrow y = 0$ or $z = x$.

i.e, either $x = y = z = 0$ or $x = y = z$.

∴ the ~~stationary~~ points are $(1,1,1)$ and $(-1,-1,-1)$

Again, we have

$$f_{xx} = 6(x+y+z) = f_{yy} = f_{zz}$$
$$f_{xy} = 6(x+y+z) - 24z = f_{yx}$$
$$f_{yz} = 6(x+y+z) - 24x = f_{zy}$$
$$f_{zx} = 6(x+y+z) - 24y = f_{xz}$$

At $(1,1,1)$

$$f_{xx} = f_{yy} = f_{zz} = 18$$
$$f_{xy} = f_{yz} = f_{zx} = -6$$

$$\therefore d^2f = f_{xx}(dx)^2 + f_{yy}(dy)^2 + f_{zz}(dz)^2 + 2f_{xy}\,dx\,dy + 2f_{yz}\,dy\,dz + 2f_{zx}\,dz\,dx.$$

$$d^2f\Big|_{(1,1,1)} = 18(dx^2 + dy^2 + dz^2) - 12(dx\,dy + dy\,dz \ldots$$

$= 6\left[3(dx^2+dy^2+dz^2) - 2(dxdy + dydz + dzdx)\right]$

$d^2f = 6\left[(dx^2+dy^2+dz^2) + (dx-dy)^2 + (dy-dz)^2 + (dz-dx)^2\right]$

which is +ve for all values of dx, dy, dz and doesnot vanish for $(dx, dy, dz) \neq (0,0,0)$.

$\therefore$ $(1,1,1)$ is a point of minima of the function i.e., f has minimum at $(1,1,1)$.

At $(-1,-1,-1)$:

$f_{xx} = f_{yy} = f_{zz} = -18$; $f_{xy} = f_{yz} = f_{zx} = 6$

$\therefore d^2f = -18\left[dx^2+dy^2+dz^2\right] + 12(dxdy + dydz + dzdx)$

$= -6\left[3(dx^2+dy^2+dz^2) - 2(dxdy + dydz + dzdx)\right]$

$= -6\left[(dx^2+dy^2+dz^2) + (dx-dy)^2 + (dy-dz)^2 + (dz-dx)^2\right]$

which is -ve for all dx, dy, dz and never vanishes.

Hence the function has maximum at $(-1,-1,-1)$.

→ S.T the following functions have a minima at the points indicated

i) $x^2+y^2+z^2+2xyz$ at $(0,0,0)$

(ii) $x^4+y^4+z^4-4xyz$ at $(1,1,1)$.

→ S.T the function

$f(x,y,z) = 2xyz - 4zx - 2yz + x^2+y^2+z^2 - 2x - 4y + 4z$

has 5 stationary points but has a minimum value only at $(1,2,0)$

→ S.T the function

$3\log(x^2+y^2+z^2) - 2x^3 - 2y^3 - 2z^3$, $(x,y,z) \neq (0,0,0)$.

has only one extreme value, $\log(3/e^2)$

→ Find a point with in a triangle such that the sum of the squares of its distances from the three vertices is a minimum.

Sol^n: Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be the vertices of the triangle and (x, y) be a point inside the triangle.

Let $f(x,y)$ denotes the sum of the squares of the distances of (x,y) from three vertices,

then

$$f(x,y) = \left[(x-x_1)^2 + (y-y_1)^2\right] + \left[(x-x_2)^2 + (y-y_2)^2\right] + \left[(x-x_3)^2 + (y-y_3)^2\right]$$

$\Rightarrow f_x = 2(x-x_1) + 2(x-x_2) + 2(x-x_3)$ & $f_y = 2(y-y_1) + 2(y-y_2) + 2(y-y_3)$

for maximum or minimum,

we have

$$f_x = 2(x-x_1) + 2(x-x_2) + 2(x-x_3) = 0$$

$$\Rightarrow 3x - (x_1 + x_2 + x_3) = 0$$

$$\Rightarrow x = \frac{x_1 + x_2 + x_3}{3}$$

Similarly

$$f_y = 2(y-y_1) + 2(y-y_2) + 2(y-y_3) = 0$$

$$\Rightarrow 3y - (y_1 + y_2 + y_3) = 0$$

$$\Rightarrow y = \frac{y_1 + y_2 + y_3}{3}$$

Also $f_{xx} = 2+2+2 = 6$

$f_{xy} = 0$ & $f_{yy} = 6$.

$f_{xx} f_{yy} - (f_{xy})^2 = (6)(6) - 0 = 36 > 0$

and $f_{xx} = 6 > 0$

$\therefore$ f is minimum when

$$x = \frac{x_1 + x_2 + x_3}{3}, \quad y = \frac{y_1 + y_2 + y_3}{3}$$

$\therefore$ The required point is $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$

→ Show that the function
$f(x,y,z) = 2xyz - 4zx - 2yz + x^2 + y^2 + z^2 - 2x - 4y + 4z$
has 5 stationary points but has a minimum value only at $(1,2,0)$.

Sol^n: Given that
$f(x,y,z) = 2xyz - 4zx - 2yz + x^2 + y^2 + z^2 - 2x - 4y + 4z$

$f_x = 2yz - 4z + 2x - 2$

$f_y = 2xz - 2z + 2y - 4$

$f_z = 2xy - 4x - 2y + 2z + 4$.

∴ The stationary points are given by

$f_x = 0 \Rightarrow 2yz - 4z + 2x - 2 = 0 \Rightarrow yz - 2z + x - 1 = 0$ —(1)

$f_y = 0 \Rightarrow 2xz - 2z + 2y - 4 = 0 \Rightarrow xz - z + y - 2 = 0$ —(2)

$f_z = 0 \Rightarrow 2xy - 4x - 2y + 2z + 4 = 0 \Rightarrow xy - 2x - y + z + 2 = 0$ —(3)

Adding ~~the last two~~ equations, we see that system ~~is~~ equivalent to

$yz - 2z + x - 1 = 0$

$zx - z + y - 2 = 0$

$zx + xy - 2x = 0 \Rightarrow x(z + y - 2) = 0$.

Thus stationary points are given by the two systems of equations

$$\left.\begin{aligned} yz - 2z + x - 1 &= 0 \\ zx - z + y - 2 &= 0 \\ x &= 0 \end{aligned}\right\}$$

&

$$\left.\begin{aligned} yz - 2z + x - 1 &= 0 \\ zx - z + y - 2 &= 0 \\ z + y - 2 &= 0 \end{aligned}\right\}$$

These give $(0,3,1)$, $(0,1,-1)$, $(1,2,0)$, $(2,1,1)$ $(2,3,-1)$ as the stationary points of the function.

Again, we have at any point (x,y,z)

$f_{xx}=2$, $f_{yy}=2$, $f_{zz}=2$;

$f_{xy}=2z$, $f_{yz}=2x-2$, $f_{zx}=2y-4=f_{xz}$
$=f_{yx}$ $\quad=f_{zy}$

for $(0,3,1)$ the matrix of the quadratic form is

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & -2 \\ 2 & -2 & 2 \end{bmatrix}.$$

Its principal minors

$$2, \quad \begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix}, \quad \begin{vmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 2 & -2 & 2 \end{vmatrix}$$

are $2, 0,$ and -32 respectively.

Thus the function is neither a maximum nor a minimum at $(0,3,1)$.

$\because f$ has minimum if principal minors are all +ve & f has maximum if principal minors are alternatingly −ve & +ve

It may similarly be shown that the function is neither a max nor a minimum at the stationary points $(0,1,-1)$, $(2,1,1)$ and $(2,3,-1)$.

At $(1,2,0)$ the ~~quadratic~~ matrix of the quadratic form is

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

the principal minors of

$$|2|,\ \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}\ \&\ \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

are 2, 4 & 8 respectively.

i.e., the principal minors are all +ve.

$\therefore f(x, y, z)$ has a minimum at $(1, 2, 0)$

---

7. Show that the function

$$3\log(x^2+y^2+z^2) - 2x^3 - 2y^3 - 2z^3, \quad (x, y, z) \neq (0,0,0).$$

has only one extreme value, $\log(3/e^2)$.

Sol:

→ Find the extreme value of xyz if $x+y+z=a$.

Sol: Let $f = xyz$.

$\phi = x+y+z-a$.

Now consider the function F of three independent variables x, y, z such that

$F = xyz + \lambda(x+y+z-a)$

where λ is a constant

$dF = (yz+\lambda)dx + (xz+\lambda)dy + (xy+\lambda)dz$.

At stationary points $dF = 0$

$\therefore f_x = 0 \Rightarrow yz+\lambda = 0$ — (1)

$f_y = 0 \Rightarrow xz+\lambda = 0$ — (2)

$f_z = 0 \Rightarrow xy+\lambda = 0$ — (3)

multiplying (1) by x, (2) by y & (3) by z and adding,

we get

$3xyz + \lambda(x+y+z) = 0$

$\Rightarrow 3xyz + \lambda(a) = 0$ $(\because x+y+z=a)$

$\Rightarrow \boxed{\lambda = \frac{-3xyz}{a}}$

From (1);

we have

$yz+\lambda = 0 \Rightarrow yz - \frac{3xyz}{a} = 0$

$\Rightarrow yz\left(1-\frac{3x}{a}\right) = 0$

$\Rightarrow 1-\frac{3x}{a} = 0$ or $yz=0$ (ignoring this)

$\Rightarrow x = \frac{a}{3}$.

from (2)

$$xz + \lambda = 0 \Rightarrow xz + \left(-\frac{3xyz}{a}\right) = 0$$

$$\Rightarrow xz\left(1 - \frac{3y}{a}\right) = 0$$

$$\Rightarrow y = \frac{a}{3}.$$

Similarly, we get $z = \frac{a}{3}$.

$\therefore$ The stationary point is $\left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)$.

$\therefore f = xyz$

$$= \left(\frac{a}{3}\right)\left(\frac{a}{3}\right)\left(\frac{a}{3}\right)$$

$$= \frac{a^3}{27} \text{ at } \left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right).$$

Hence $f = xyz = \frac{a^3}{27}$ —— (4)

Now $d^2F = d(dF)$

$$= d\left[(yz+\lambda)dx + (xz+\lambda)dy + (xy+\lambda)dz\right]$$

$$= \left[(yz+\lambda)dx^2 + z\,dy + y\,dz\right]dx + \left[(xz+\lambda)dy^2 + z\,dx + x\,dz\right]dy + \left[(xy+\lambda)dz^2 + (y\,dx + x\,dy)\right]dz.$$

$$= 2(z\,dx\,dy + x\,dy\,dz + y\,dx\,dz),$$

(as $f_x = 0, f_y = 0, f_z = 0$)

Here $f_{xx} = 0$, $f_{yy} = 0$, $f_{zz} = 0$

$$\cancel{\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 0 & f_{xy} \\ f_{yx} & 0 \end{vmatrix}}$$

and $$\cancel{\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} 0 & f_{xy} & f_{xz} \\ f_{yx} & 0 & f_{yz} \\ f_{zx} & f_{zy} & 0 \end{vmatrix}}$$

~~we require further investigation.~~

Here $f_{xx} = 0$.

$\therefore$ we require further investigation.

Treating z as function of x and y,

we get from

$$f(x,y,z) = xyz - \frac{a^3}{27} = 0$$

$$yz + xy\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{z}{x}$$

Similarly $\frac{\partial z}{\partial y} = -\frac{z}{y}$.

Also $$\frac{\partial^2 z}{\partial x^2} = -\frac{\left[x\frac{\partial z}{\partial x} - z\right]}{x^2}$$

$$= -\frac{\left[x\left(-\frac{z}{x}\right) - z\right]}{x^2}$$

$$= \frac{2z}{x^2}$$

Similarly $\frac{\partial^2 z}{\partial y^2} = \frac{2z}{y^2}$, $$\frac{\partial^2 z}{\partial x \partial y} = -\frac{\left(y\frac{\partial z}{\partial x} - z(0)\right)}{y^2}$$

$$= -\frac{\left[y\left(-\frac{z}{x}\right) - 0\right]}{y^2}$$

$$= \frac{z}{xy}.$$

At $\left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)$

$$z_{xx} = \frac{2\left(\frac{a}{3}\right)}{\left(\frac{a}{3}\right)^2} = \frac{2\times 3}{a} = \frac{6}{a} > 0 \text{ if } a > 0$$

$$z_{yy} = \frac{6}{a} > 0$$

$$z_{xy} = \frac{a/3}{(a/3)^2} = \frac{3}{a} > 0.$$

$z_{xx} > 0$ and $z_{xx}z_{yy} - z_{xy}^2 = \left(\frac{6}{a}\right)\left(\frac{6}{a}\right) - \left(\frac{3}{a}\right)^2$

$$= \frac{36}{a^2} - \frac{9}{a^2}$$

$$= \frac{27}{a^2} > 0.$$

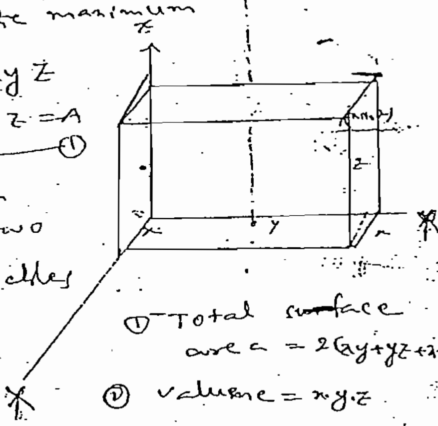
$\therefore$ $f(x,y,z)$ has a minimum value at $\left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)$ and the minimum value is 3abc.

* Lagrange's method of undetermined multipliers (for several independent variables)

Lagrange's method of multipliers, enable us to locate the stationary points when the variables are not free but are subject to some additional conditions.

Suppose we want to construct a closed box in the form of a parallelopiped of maximum volume using a piece of tin of area 'A'. Let x, y, z denote the length, width and height of the box, respectively. Then the problem reduces to finding the maximum of the function $f(x,y,z) = xyz$

given that $2xy + 2xz + 2yz = A$ —— (1)



(1) Total surface area $= 2(xy+yz+xz)$

(2) volume $= x.y.z$

We shall now discuss such problems for functions of two variables, where the variables satisfy some side conditions as in (1). That is, we will discuss a method to find out the maximum and minimum values of a function $z = f(x,y)$, given that x and y are connected by an equation $g(x,y) = 0$.

If we eliminate one variable from the equation $z = f(x,y)$ with the help of the relation $g(x,y) = 0$, then, z becomes a function of one variable. Then we can easily find its extreme values. So, the problem reduces to finding the maximum and minimum values for a function of one variable.

For example:

Find the extreme values of the function $f(x,y) = x^2+2y^2-x$ on the unit circle $x^2+y^2=1$.

Sol Given that $f(x,y) = x^2+2y^2-x$ — (1)

subject to condition

$x^2+y^2=1$ — (2)

$\Rightarrow \boxed{y^2 = 1-x^2}$

$\therefore f(x,y) = f(x, \sqrt{1-x^2})$

$= x^2+2(1-x^2)-x$ (which is clearly in one variable)

$= -x^2+2-x$

$= g(x)$ say

i.e $g(x) = -x^2+2-x$ — (3)

Now we shall find out the points of extrema for $g(x)$.

For this we find $g'(x)$ and equating it to zero.

$g'(x) = -2x-1$

$\Rightarrow -2x-1=0$

$\Rightarrow \boxed{x = -1/2}$ is a stationary point of $g(x)$.

Now we check whether it is a maximum or minimum point.

For this, $g''(x) = -2$

$\therefore g''(-1/2) = -2 < 0$

$\therefore$ $g(x)$ has maximum at $x = -1/2$.

$g(-1/2) = -\frac{1}{4}+2+\frac{1}{2}$

$= \frac{9}{4}$, which is the reqd maximum value of the function $f(x,y)$ in the unit circle.

(or)

$g(x)$ has maximum value at $x = -\frac{1}{2}$

so that

from ②, $y^2 = 1 - \frac{1}{4} = \frac{3}{4}$

$$\therefore \boxed{y = \pm \frac{\sqrt{3}}{2}}$$

Therefore, we conclude that the function has a maximum at two points $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

Also $f(x,y) = f\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$$= \frac{1}{4} + \frac{3}{2} + \frac{1}{2}$$

$$= \frac{9}{4}$$

$$= f\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right).$$

Thus, the maximum value of the function on the unit circle is $\frac{9}{4}$.

Note :- we must have found this example quite easy to follow. But it is not always feasible to use this procedure. The reduction of the given function to a function of one variable using the given constraints might prove to be cumbersome or sometimes might not possible at all.

we now present an alternative method which is often more convenient. This method is known as the method of Lagrange's multipliers:

Suppose we want to maximize or minimize a function $Z = f(x,y)$ subject to the condition $g(x,y) = 0$.

Theoretically, Z is a function of a single variable say x and the extreme values $\frac{dZ}{dx} = 0$

but $\frac{dz}{dx} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$

$$\Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \quad \text{——— (1)}$$

from the relation $g(x,y) = 0$, we find that at the extreme ~~value~~ have $\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}\cdot\frac{dy}{dx} = 0$. ——— (2)

Multiplying eqn (2) by an undetermined multiplier 'λ' and adding this to equation (1), we get

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x}\right) + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y}\right)\frac{dy}{dx} = 0 \quad \text{——— (3)}$$

choosing λ so that the coefficient of $\frac{dy}{dx} = 0$ in (3), Hence at the points of extrema, we have

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x} = 0 \quad \text{——— (4)}$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y} = 0 \quad \text{——— (5)}$$

$$g(x,y) = 0 \quad \text{——— (6)}$$

- from these equations we can determine the three unknowns ~~x, y, λ~~.
- The values of x, y give us the co-ordinates of the stationary points
- The role of λ is over and we don't need it any more.
- we may add here that each stationary point so determined need to be a max. or min.
- sometimes, we can determine their nature by simple observation of equation $z = f(x,y)$.
- In some cases we can apply the second derivative test, by eliminating the dependent variable.

— In fact, we can observe that equations ④, ⑤ and ⑥ are obtained by equating the partial derivatives of the function

$F(x,y,\lambda) = f(x,y) + \lambda g(x,y)$ to zero, treating x, y and λ as independent variables.

~~Suppose~~ we are given the function $f(x,y)$, whose extrema are to be found subject to the constraint $g(x,y) = 0$. We form the auxiliary function

$F(x,y,\lambda) = f(x,y) + \lambda g(x,y)$ ——— ⑦

where λ is to be determined.

Then we find the three partial derivatives of $F(x,y,\lambda)$ and equate these to zero.

Then we solve three equations. The values of (x,y) thus obtained are the stationary points of the given function under the given constraint.

The number λ is called Lagrange's multiplier.

For example:

Find the largest and the smallest values of $f(x,y) = x+2y$ on the circle $x^2+y^2=1$.

Sol. $f(x,y) = x+2y$ ——— ①

and $g(x,y) = x^2+y^2-1$ ——— ②

Now the auxiliary function is

$F(x,y,\lambda) = (x+2y) + \lambda(x^2+y^2-1)$ ——— ⑦

partially diff ⑦ w.r.t x, y and λ, and equating to zero, we get

$$1+\lambda(2x)=0 \quad \text{--- (4)}$$
$$2+\lambda(2y)=0 \quad \text{--- (5)}$$
$$x^2+y^2=1 \quad \text{--- (6)}$$

solving (4) and (5), we get

$$x=-\frac{1}{2\lambda},\ y=-\frac{1}{\lambda} \quad \text{--- (7)}$$

$$(6) \equiv \left(\frac{-1}{2\lambda}\right)^2+\left(\frac{-1}{2\lambda}\right)^2=1$$

$$\Rightarrow \lambda^2=\frac{5}{4}$$

$$\boxed{\lambda=\pm\frac{\sqrt{5}}{2}} \quad \text{It's given}$$

from (7), $x=\frac{-1}{\sqrt{5}}$

$y=\frac{-2}{\sqrt{5}}$ $\left(\because \lambda=+\frac{\sqrt{5}}{2}\right)$.

and $f(x,y)=-\frac{5}{\sqrt{5}}=-\sqrt{5}$

from (7), $x=\frac{1}{\sqrt{5}},\ y=\frac{2}{\sqrt{5}}$ $\left(\because \lambda=-\frac{\sqrt{5}}{2}\right)$

and $\boxed{f(x,y)=+\sqrt{5}.}$

Thus, the stationary points are $\left(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}\right)$ and $\left(\frac{-1}{\sqrt{5}},\frac{-2}{\sqrt{5}}\right)$.

Since $f\left(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}\right)=\sqrt{5}$ and $f\left(\frac{-1}{\sqrt{5}},\frac{-2}{\sqrt{5}}\right)=-\sqrt{5}$

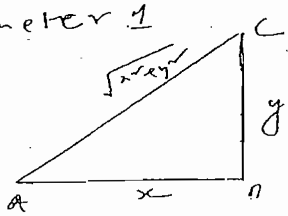
we get the largest value is $\sqrt{5}$ and smallest value is $-\sqrt{5}$.

HW → (1) Find the extreme values of the function $f(x,y)=xy$ on the surface $g(x,y)$ where $g(x,y)=\frac{x^2}{8}+\frac{y^2}{2}-1=0$.

Ans: Min = -2, Max = $+2$

→ find the right angled triangle of perimeter 1 with largest area.

Sol Suppose ABC is a right angled triangle with perimeter 1



Let the sides of triangle be $x, y, \sqrt{x^2+y^2}$.

then $f(x,y) = \text{area of } \Delta^{le} ABC$

$= \frac{1}{2}xy$

and the perimeter of the triangle ABC

$= x+y+\sqrt{x^2+y^2}$

but given that

$x+y+\sqrt{x^2+y^2} = 1$.

Now we have to find the maximum of

$f(x,y) = \frac{1}{2}xy$ —— (1)

subject to the condition

$g(x,y) = x+y+\sqrt{x^2+y^2}-1 = 0$. —— (2)

Let us form the system of the equations for this f and g. we get

$\frac{1}{2}y + \lambda\left[1+\frac{x}{\sqrt{x^2+y^2}}\right] = 0$ —— (3)

$\frac{1}{2}x + \lambda\left[1+\frac{y}{\sqrt{x^2+y^2}}\right] = 0$ —— (4)

$x+y+\sqrt{x^2+y^2}-1 = 0$ —— (5)

from the first two equations, we have

$$\frac{\frac{1}{2}y}{1+\frac{x}{\sqrt{x^2+y^2}}} = \frac{\frac{1}{2}x}{1+\frac{y}{\sqrt{x^2+y^2}}}$$

(or)

$$\frac{y}{\sqrt{x^2+y^2}+x} = \frac{x}{\sqrt{x^2+y^2}+y}$$

(or)

$$\frac{y}{1-y} = \frac{x}{1-x}$$

$$\Rightarrow y-xy = x-xy$$

$$\Rightarrow \boxed{x=y}$$

$\therefore$ The sides are x, x and $\sqrt{2}x$ s.t

$$x+x+\sqrt{2}x = 1$$

i.e $(\sqrt{2}+\sqrt{2})x = 1$

$$\Rightarrow \boxed{x = \frac{1}{2+\sqrt{2}}}$$

Thus, the reqd sides are

$\frac{1}{2+\sqrt{2}}$, $\frac{1}{2+\sqrt{2}}$ and $\frac{1}{1+\sqrt{2}}$.

→ To find the stationary points of the function $f(x_1, x_2, \ldots x_n, u_1, u_2, \ldots u_m)$ of $n+m$ variables which —(1) are connected by the equations

$$\phi_r(x_1, x_2, \ldots x_n, u_1, u_2, \ldots u_m) = 0, \quad r = 1, 2, 3, \ldots m. \quad \text{—(2)}$$

If the m variables $u_1, u_2, \ldots u_m$ are determinate as functions of $x_1, x_2, \ldots x_n$ from the system of 'm' equation (2), then f can be regarded as function of 'n' independent variables $x_1, x_2, \ldots x_n$.

For stationary values, $df = 0$

$$\therefore 0 = df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \cdots + \frac{\partial f}{\partial x_n}dx_n + \frac{\partial f}{\partial u_1}du_1 + \cdots + \frac{\partial f}{\partial u_m}du_m. \quad \text{—(3)}$$

Differentiating equations (2), we get

$$\left.\begin{aligned}
&\frac{\partial \phi_1}{\partial x_1}dx_1 + \cdots + \frac{\partial \phi_1}{\partial x_n}dx_n + \frac{\partial \phi_1}{\partial u_1}du_1 + \cdots + \frac{\partial \phi_1}{\partial u_m}du_m = 0\\
&\frac{\partial \phi_2}{\partial x_1}dx_1 + \cdots + \frac{\partial \phi_2}{\partial x_n}dx_n + \frac{\partial \phi_2}{\partial u_1}du_1 + \cdots + \frac{\partial \phi_2}{\partial u_m}du_m = 0\\
&\vdots\\
&\frac{\partial \phi_m}{\partial x_1}dx_1 + \cdots + \frac{\partial \phi_m}{\partial x_n}dx_n + \frac{\partial \phi_m}{\partial u_1}du_1 + \cdots + \frac{\partial \phi_m}{\partial u_m}du_m = 0
\end{aligned}\right\} \quad \text{—(4)}$$

Multiplying the equations (4) by $\lambda_1, \lambda_2, \ldots \lambda_m$ respectively and adding to the eqn(3) we get

$$0 = df = \left(\frac{\partial f}{\partial x_1} + \sum \lambda_r \frac{\partial \phi_r}{\partial x_1}\right)dx_1 + \cdots + \left(\frac{\partial f}{\partial x_n} + \sum \lambda_r \frac{\partial \phi_r}{\partial x_n}\right)dx_n.$$
$$+ \left(\frac{\partial f}{\partial u_1} + \sum \lambda_r \frac{\partial \phi_r}{\partial u_1}\right)du_1 + \cdots + \left(\frac{\partial f}{\partial u_m} + \sum \lambda_r \frac{\partial \phi_r}{\partial u_m}\right)du_m \quad \text{—(5)}$$

$r = 1, 2, \ldots m.$

Let the 'm' multipliers $\lambda_1, \lambda_2, \ldots \lambda_m$ be so chosen

that the coefficients of m differentials $du_1, du_2, \ldots du_m$ all vanish.

i.e, $$\left.\begin{aligned} &\frac{\partial f}{\partial u_1} + \Sigma \lambda_r \frac{\partial \phi_r}{\partial u_1} = 0, \\ &\frac{\partial f}{\partial u_2} + \Sigma \lambda_r \frac{\partial \phi_r}{\partial u_2} = 0 \\ &\vdots \\ &\frac{\partial f}{\partial u_m} + \Sigma \lambda_r \frac{\partial \phi_r}{\partial u_m} = 0 \end{aligned}\right\} \text{—} (6)$$

then eqn (5) becomes

$$0 = df = \left(\frac{\partial f}{\partial x_1} + \Sigma \lambda_r \frac{\partial \phi_r}{\partial x_1}\right) dx_1 + \cdots\cdots + \left(\frac{\partial f}{\partial x_n} + \Sigma \lambda_r \frac{\partial \phi_r}{\partial x_n}\right) dx_n$$

So that the differential df is expressed interms of the differentials of Independent variables only.

Hence $$\left.\begin{aligned} &\frac{\partial f}{\partial x_1} + \Sigma \lambda_r \frac{\partial \phi_r}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} + \Sigma \lambda_r \frac{\partial \phi_r}{\partial x_2} = 0, \cdots \\ &\frac{\partial f}{\partial x_n} + \Sigma \lambda_r \frac{\partial \phi_r}{\partial x_n} = 0. \end{aligned}\right\} \text{—} (7)$$

Equations (2), (6), (7) form a system of $m+m+n = 2m+n$ equations which may be simultaneously solved to determine the 'm' multipliers $\lambda_1, \lambda_2, \ldots \lambda_m$ and the $n+m$ coordinates $x_1, x_2, \ldots x_n, u_1, u_2, \ldots u_m$ of the stationary points of f.

Important Rules

For practical purposes, the process of obtaining equations (6) and (7) of the above, may be put in a precise form as follows

Define a function

$$F = f + \lambda_1 \phi_1 + \lambda_2 \phi_2 + \cdots + \lambda_m \phi_m.$$

and consider all the variables $x_1, x_2, \ldots x_n, u_1, u_2, \ldots u_m$

as independent.

At a stationary point of F, $dF = 0$.

$$\therefore 0 = dF = \frac{\partial F}{\partial x_1}dx_1 + \frac{\partial F}{\partial x_2}dx_2 + \cdots + \frac{\partial F}{\partial x_n}dx_n + \frac{\partial F}{\partial u_1}du_1 + \cdots + \frac{\partial F}{\partial u_m}du_m.$$

$$\therefore \frac{\partial F}{\partial x_1} = 0,\ \frac{\partial F}{\partial x_2} = 0,\ \cdots \frac{\partial F}{\partial x_n} = 0,\ \frac{\partial F}{\partial u_1} = 0,\ \cdots \frac{\partial F}{\partial u_m} = 0$$

which are same as equations (6) & (7).

$\therefore$ The stationary points of f may be found by determining the stationary points of the function F, where $F = f + \lambda_1\phi_1 + \lambda_2\phi_2 + \cdots + \lambda_m\phi_m$. and considering all the variables as independent variable.

→ A stationary point will be an extreme point of f ~~if~~ d^2F keeps the same sign, and will be maxima or minima according as d^2F is $-ve$ or $+ve$.

→ Find the shortest distance from the origin to the hyperbola $x^2 + 8xy + 7y^2 = 225$, $z = 0$.

Sol^n: We have to find the minimum value of x^2+y^2 (the square of the distance from the origin to any point in the xy plane) subject to the constraint,

$$x^2 + 8xy + 7y^2 = 225.$$

Let $f = x^2+y^2$; $\phi = x^2 + 8xy + 7y^2 - 225 = 0$

Consider the function

$$F = x^2+y^2+\lambda(x^2 + 8xy + 7y^2 - 225).$$

where x, y are independent variables and λ is a constant.

$df = (2x + 2x\lambda + 8y\lambda)\,dx + (2y + 8x\lambda + 14y\lambda)\,dy$

$\therefore \frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ $\quad \left(\because df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right)$

$\Rightarrow 2(1+\lambda)x + 8y\lambda = 0$; $\Rightarrow 2(1+7\lambda)y + 8\lambda x = 0$

$\Rightarrow (1+\lambda)x + 4y\lambda = 0$; —(1) $\Rightarrow 4\lambda x + (1+7\lambda)y = 0$ (2)

Now eliminating x and y from (1) & (2)

we get

$$\begin{vmatrix} 1+\lambda & 4y \\ 4\lambda & 1+7\lambda \end{vmatrix} = 0$$

$\Rightarrow (1+\lambda)(1+7\lambda) - 16\lambda^2 = 0$

$\Rightarrow 1 + 8\lambda + 7\lambda^2 - 16\lambda^2 = 0$

$\Rightarrow -9\lambda^2 + 8\lambda + 1 = 0$

$\Rightarrow 9\lambda^2 - 8\lambda - 1 = 0$

$\Rightarrow 9\lambda^2 - 9\lambda + \lambda - 1 = 0$

$\Rightarrow 9\lambda(\lambda - 1) + 1(\lambda - 1) = 0$

$\Rightarrow (9\lambda + 1)(\lambda - 1) = 0$

$\Rightarrow \lambda = -\frac{1}{9}$; $\lambda = 1$

If $\lambda = 1$, from(1) $x = -2y$.

Now from $x^2 + 8xy + 7y^2 = 225$

we get $y^2 = -45$, for which no real solution exists.

If $\lambda = -\frac{1}{9}$, from(1)

we get $y = 2x$.

Now from $x^2 + 8xy + 7y^2 = 225$

we get $x^2 = 5$, $y^2 = 20$

$\therefore x^2 + y^2 = 25$.

Now $d^2F = d(dF)$

$$= d\left[\left(2x + 2x\lambda + 8y\lambda\right)dx + \left(2y + 8x\lambda + 14y\lambda\right)dy\right]$$

$$= \left[(2+2\lambda)\,dx + 8\lambda dy\right]dx + \left[8\lambda dx + (2+14\lambda)dy\right]dy$$
$$+ (2x+2x\lambda+8y\lambda)d^2x + (2y+8x\lambda+14y\lambda)d^2y$$

$$= 2(1+\lambda)(dx)^2 + 2(1+7\lambda)(dy)^2 + 16\lambda\,dx\,dy$$

$\left(\because \frac{\partial f}{\partial x} = 0\right.$ i.e., $2x+2x\lambda+8y\lambda=0$ and $\frac{\partial f}{\partial y}=0 \Rightarrow 2y+8x\lambda+14y\lambda=0$

At $\lambda = -\frac{1}{9}$, $d^2F = \frac{16}{9}(dx)^2 - \frac{16}{9}dx\,dy + \frac{4}{9}(dy)^2$

$$= \frac{4}{9}\left[4(dx)^2 - 4\,dx\,dy + (dy)^2\right]$$

$$= \frac{4}{9}(2dx - dy)^2$$

> 0, and cannot vanish because $(dx, dy) \neq (0,0)$

Hence the function x^2+y^2 has a minimum value 25.

→ Find the maximum and minimum of $x^2+y^2+z^2$ subject to the conditions $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$ and $z = x+y$.

Solⁿ Let $f = x^2+y^2+z^2$

$$\phi_1 = \frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} - 1 = 0$$

and $\phi_2 = x+y-z = 0$

Consider a function F of independent variables x, y, z where:

$$F = x^2+y^2+z^2 + \lambda_1\left(\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} - 1\right) + \lambda_2(x+y-z)$$

$$\therefore dF = \left(2x + \frac{x}{2}\lambda_1 + \lambda_2\right)dx + \left(2y + \frac{2y}{5}\lambda_1 + \lambda_2\right)dy + \left(2z + \frac{2z}{25}\lambda_1 - \lambda_2\right)dz$$

$[dF = F_x dx + F_y dy + F_z dz$

As x, y, z are independent variables,

$\therefore \frac{\partial f}{\partial x} = 0 \Rightarrow 2x + \frac{x}{2}\lambda_1 + \lambda_2 = 0$

$\frac{\partial f}{\partial y} = 0 \Rightarrow 2y + \frac{2y}{5}\lambda_1 + \lambda_2 = 0$

$\frac{\partial f}{\partial z} = 0 \Rightarrow 2z + \frac{2z}{25}\lambda_1 - \lambda_2 = 0$

$\therefore x = \frac{-2\lambda_2}{\lambda_1+4}, \; y = \frac{-5\lambda_2}{2\lambda_1+10}, \; z = \frac{25\lambda_2}{2\lambda_1+50}$ —— (1)

Substituting in $x+y=z$, we get

$$-\frac{2\lambda_2}{\lambda_1+4} + \frac{(-5\lambda_2)}{5\lambda_1+10} - \frac{25\lambda_2}{2\lambda_1+50} = 0$$

$$\lambda_2\left[\frac{2}{\lambda_1+4} + \frac{5}{5\lambda_1+10} + \frac{25}{2\lambda_1+50}\right] = 0$$

if $\lambda_2 \neq 0$, $\frac{2}{\lambda_1+4} + \frac{5}{5\lambda_1+10} + \frac{25}{2\lambda_1+50} = 0$ —— (2)

if $\lambda_2 = 0$

then from (1)

$x = 0, \; y = 0, \; z = 0$, but $(x,y,z) = (0,0,0)$ doesnot satisfy the other condition of the constraint.

$\therefore$ from (2), $17\lambda_1^2 + 245\lambda_1 + 750 = 0$

$\Rightarrow \lambda_1 = -10, \; \lambda_1 = -75/17$

For $\lambda_1 = -10$, from (1)

$x = \frac{1}{3}\lambda_2, \; y = \frac{1}{2}\lambda_2, \; z = \frac{5}{6}\lambda_2$ —— (3)

Now substituting (3) in $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$

we get $\lambda_2^2\left[\frac{1}{36} + \frac{1}{20} + \frac{1}{36}\right] = 1$

$\frac{19\lambda_2^2}{180} = 1 \Rightarrow \lambda_2^2 = \frac{180}{19}$ (or) $\lambda_2 = \pm 6\sqrt{}$

putting $\lambda_2 = \pm 6\sqrt{5/19}$ in (3)

from (3) The corresponding stationary points are

$\left(2\sqrt{5/19},\ 3\sqrt{5/19},\ 5\sqrt{5/19}\right)$ and $\left(-2\sqrt{5/19}, -3\sqrt{5/19}, -5\sqrt{5/19}\right)$

The value of $x^2+y^2+z^2$ corresponding to these points is 10.

for $\lambda_1 = -75/17$

from (1) $x = \frac{34}{7}\lambda_2,\quad y = -\frac{17}{4}\lambda_2,\quad z = \frac{17}{28}\lambda_2$ —— (2)

which on substitution in $\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1$ give

$$\lambda_2 = \frac{\pm 140}{17\sqrt{646}}$$

Substituting $\lambda_2 = \frac{\pm 140}{17\sqrt{646}}$ in (2)

Then the corresponding stationary points are

$\left(\frac{40}{\sqrt{646}}, \frac{-35}{\sqrt{646}}, \frac{5}{\sqrt{646}}\right)$ and $\left(-\frac{40}{\sqrt{646}}, \frac{35}{\sqrt{646}}, \frac{-5}{\sqrt{646}}\right)$

The value of $x^2+y^2+z^2$ corresponding to these points is 75/17.

$\therefore$ the maximum value is 10 and the minimum value is 75/17.

Note:

[1]. We have not theoretically established the existence of maximum or minimum value. We have simply shown that of all possible values, 10 is maximum and 75/17 the minimum.

[2]. Using constraint conditions, $dz = dx + dy$; $\frac{x}{4}dx + \frac{y}{5}dy + \frac{z}{25}$ so that ~~and~~ dz, dy and consequently d^2F may be expressed in terms of dx (or dx^2) alone. It can, then, be easily verified that 10 is a maximum value and 75/17 the minimum

→ Determine the extreme values of $bcx + cay + abz$ subject to the conditions $xyz = abc$ where $a>0, b>0, c>0$

Soln Let $f = bcx + cay + abz$.
$\phi = xyz - abc$.

Now consider a function f of three independent variables x, y, z, where

$f = bcx + cay + abz + \lambda(xyz - abc)$
where λ is constant.

$df = (bc + yz\lambda)dx + (ca + xz\lambda)dy + (ab + \lambda xy)dz$.
$(\because df = f_x dx + f_y dy + f_z dz)$

As x, y, z are independent variables.

$\therefore \frac{\partial f}{\partial x} = 0 \Rightarrow bc + yz\lambda = 0$
$\Rightarrow \lambda yz = -bc$ —— (1)

$\frac{\partial f}{\partial y} = 0 \Rightarrow ca + xz\lambda = 0$
$\Rightarrow xz\lambda = -ca$ —— (2)

$\frac{\partial f}{\partial z} = 0 \Rightarrow ab + \lambda xy = 0$
$\Rightarrow \lambda xy = -ab$ —— (3)

Multiplying (1), (2), (3)
$\lambda^3 x^2 y^2 z^2 = -a^2 b^2 c^2$
$\lambda^3 (abc)^2 = -a^2 b^2 c^2$ $(\because xyz = abc)$
$\Rightarrow \lambda^3 = -1$
$\Rightarrow \lambda = -1$; ignoring the imaginary values of λ.

Sub $\lambda = -1$ in (1), (2) & (3)
$\therefore$ $\Rightarrow yz = bc$
$xz = ca$
$xy = ab$

Solving we get the stationary point as (a, b, c).
i.e. $x = a, y = b, z = c$

$\therefore f = bcx + cay + abz$ at (a, b, c) $= abc + abc + abc$
$= 3abc$

Now $d^2F = d(dF) = d[(bc+yz\lambda)dx + (ac+xz\lambda)dy + (ab+\lambda xy)dz]$

$\therefore d^2F = -2[y\,dx\,dz + z\,dx\,dy + x\,dy\,dz]$ at $\lambda = -1$

$\because$ the coefficients of dx^2, dy^2, dz^2 i.e, $f_{xx}=0$, $f_{yy}=0$, $f_{zz}=0$

Here $F_{xx}=0$, $F_{yy}=0$, $F_{zz}=0$

$\begin{vmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{vmatrix} = 0$

and $\begin{vmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{vmatrix} = \begin{vmatrix} 0 & F_{xy} & F_{xz} \\ F_{yx} & 0 & F_{yz} \\ F_{zx} & F_{zy} & 0 \end{vmatrix} = 0$

$\therefore d^2F = 0$.

we require further investigation.

Treating z as function of x and y, we get from $\phi(x,y,z) = xyz - abc = 0$

$yz + xy\frac{\partial z}{\partial x} = 0$ (or) $\frac{\partial z}{\partial x} = -\frac{z}{x}$

Similarly $\frac{\partial z}{\partial y} = -\frac{z}{y}$ — (62)

Also, $\frac{\partial^2 z}{\partial x^2} = -\frac{\left[x\frac{\partial z}{\partial x} - z\right]}{x^2} = \frac{2z}{x^2}$

Similarly $\frac{\partial^2 z}{\partial y^2} = \frac{2z}{y^2}$, $\frac{\partial^2 z}{\partial x \partial y} = 0$

Hence at (a,b,c), we have $z_{xx} = \frac{2z}{x^2} = \frac{2c}{a^2} > 0$; $z_{yy} = \frac{2}{b}$

$z_{xy} = 0$

$z_{xx} > 0$ and $z_{xx}z_{yy} - z_{xy}^2 = \frac{2c}{a^2}\cdot\frac{2c}{b^2} - 0 > 0$

$\therefore f(x,y,z)$ has a minimum value at (a,b,c) and the minimum value is $3abc$

→ P.T the volume of the greatest rectangular parallelopiped, that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, is $\frac{8abc}{3\sqrt{3}}$.

Soln. If $P(x,y,z)$ is the vertex of a parallelopiped inscribed in $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, then the part that lies in the same quadrant as P will have volume xyz. for the largest parallelopiped such parts in all the quadrants must be similar and ~~hen~~ hence the volume will be $8xyz$.

So, we are to find the maximum value of $8xyz$ subject to the conditions.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,\ x>0,\ y>0,\ z>0 \quad (1)$$

Let $f = 8xyz$
and $\phi = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$

Let us consider a function F of three independent variables x, y, z. where

$$F = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

$$\therefore df = \left(8yz + \frac{2x\lambda}{a^2}\right)dx + \left(8xz + \frac{2y\lambda}{b^2}\right)dy + \left(8xy + \frac{2\lambda z}{c^2}\right)$$

~~Atcg~~ At stationary points

$$\left.\begin{aligned} f_x = 0 &\Rightarrow 8yz + \frac{2x\lambda}{a^2} = 0 \\ f_y = 0 &\Rightarrow 8xz + \frac{2y\lambda}{b^2} = 0 \\ f_z = 0 &\Rightarrow 8xy + \frac{2z\lambda}{c^2} = 0 \end{aligned}\right\} \quad \text{--- (2)}$$

Multiplying (2) by x, y, z respectively and adding

$$24xyz + 2\lambda\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right] = 0$$

$$\Rightarrow 24xyz + 2\lambda(1) = 0 \qquad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$$

$$\Rightarrow 12xyz + \lambda = 0$$

$$\Rightarrow \lambda = -12xyz$$

$\therefore$ from (2)

$$x = \frac{a}{\sqrt{3}},\ y = \frac{b}{\sqrt{2}},\ c = c/\sqrt{3}$$

and so $\lambda = -12\left(\frac{a}{\sqrt{3}}\right)\left(\frac{b}{\sqrt{2}}\right)\left(\frac{c}{\sqrt{3}}\right)$

$$= \frac{-12abc}{3\sqrt{2}} = \frac{-4abc}{\sqrt{3}}$$

Again

$$d^2F = 2\lambda\left(\frac{dx^2}{a^2} + \frac{dy^2}{b^2} + \frac{dz^2}{c^2}\right) + 16z\,dx\,dy$$

$$+ 16x\,dy\,dz + 16y\,dx\,dz \qquad \left(\because \text{the coefficient of } dx, dy, \ldots\ f_x = 0, f_y = 0, f_z = 0\right)$$

$$d^2F = -\frac{8abc}{\sqrt{3}}\sum \frac{1}{a^2}dx^2 + \frac{16}{\sqrt{3}}\sum c\,dx\,dy \quad \text{--- (3)}$$

Now from (1)

$$x\frac{dx}{a^2} + y\frac{dy}{b^2} + z\frac{dz}{b^2} = 0$$

at $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{2}}, \frac{c}{\sqrt{3}}\right)$

$$\frac{a}{\sqrt{3}}\frac{dx}{a^2} + \frac{b\,dy}{\sqrt{3}\,b^2} + \frac{c\,dz}{\sqrt{3}\,c^2} = 0$$

$$\Rightarrow \frac{dx}{a} + \frac{dy}{b} + \frac{dz}{c} = 0 \quad \text{—— (4)}$$

Squaring, we get

$$\sum \frac{dx^2}{a^2} + 2\sum \frac{dx\,dy}{ab} = 0$$

(or) $$\sum \frac{dx^2}{a^2} + 2\times\frac{1}{abc}\sum c\,dx\,dy = 0$$

$$\Rightarrow abc\sum dx^2 + 2\sum c\,dx\,dy = 0$$

$$\Rightarrow \boxed{abc\sum \frac{dx^2}{a^2} = -2\sum c\,dx\,dy} \quad (5)$$

from (3) $$d^2F = -\frac{8abc}{\sqrt{3}}\sum\frac{dx^2}{a^2} + \frac{16}{\sqrt{3}}\left(\frac{abc}{2}\right)\sum\frac{dx^2}{a^2}$$ ($\because$ from

$$d^2F = -\frac{8abc}{\sqrt{3}}\sum\frac{dx^2}{a^2} - \frac{8abc}{\sqrt{3}}\sum\frac{dx^2}{a^2}$$

$$= -\frac{16}{\sqrt{3}}abc\sum\frac{dx^2}{a^2}$$

$$< 0$$

Hence $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{2}}, \frac{c}{\sqrt{3}}\right)$ is a point of maximum and the maximum value of $8xyz$ is $\frac{8abc}{3\sqrt{3}}$

→ S.T that the length of axes of the section of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ by the plane $lx+my+nz=0$ are the roots of the quadratic in r^2,

$$\frac{l^2a^2}{r^2-a^2} + \frac{m^2b^2}{r^2-b^2} + \frac{n^2c^2}{r^2-c^2} = 0$$

Sol^n: we have to find the stationary values of the function r^2, where $r^2 = x^2+y^2+z^2$ subject to the two equations of condition

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \text{---(1)}$$

$$lx+my+nz=0 \quad \text{---(2)}$$

Let us consider a function F of independent variables x, y, z.

Let $f = x^2+y^2+z^2$.

$$\phi_1 = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$$

$$\phi_2 = lx+my+nz=0$$

$$\therefore F = (x^2+y^2+z^2)+\lambda_1\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)+2\lambda_2(lx+my+nz)$$

$$\therefore dF = \left(2x+\frac{2x\lambda_1}{a^2}+2\lambda_2 l\right)dx + \left(2y+\frac{2y\lambda_1}{b^2}+2\lambda_2 m\right)dy + \left(2z+\frac{2z\lambda_1}{c^2}+2\lambda_2 n\right)dz$$

At stationary points

$$x+\frac{x}{a^2}\lambda_1+l\lambda_2=0,\quad y+\frac{y}{b^2}\lambda_1+m\lambda_2=0$$

$$z+\frac{z}{c^2}\lambda_1+n\lambda_2=0 \quad \text{---(3)}$$

Multiplying by x, y, z respectively and adding, we get

$$\lambda_1\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)+x^2+y^2+z^2+\lambda_2(lx+my+nz)=0$$

$$\Rightarrow \lambda_1(1)+x^2+y^2+z^2+\lambda_2(0)=0 \quad (\because \text{from (1) \& (2)})$$

$$\Rightarrow x^2+y^2+z^2+\lambda_1=0$$

$$\Rightarrow \lambda_1 = -(x^2+y^2+z^2)$$

$$\Rightarrow \lambda_1 = -r^2$$

$\therefore$ from (3)

$$x=\frac{a^2 l\lambda_2}{r^2-a^2};\quad y=\frac{b^2 m\lambda_2}{r^2-b^2};\quad z=\frac{c^2 n\lambda_2}{r^2-c^2}$$

Substituting these values in $lx+my+nz=0$

$$\Rightarrow \lambda_2 \left\{ \frac{a^2l^2}{r^2-a^2} + \frac{b^2m^2}{r^2-b^2} + \frac{c^2n^2}{r^2-c^2} \right\} = 0$$

and since $\lambda_2 \neq 0$,

we get the quadratic in r^2 giving the stationary values:

$$\frac{a^2l^2}{r^2-a^2} + \frac{b^2m^2}{r^2-b^2} + \frac{c^2n^2}{r^2-c^2} = 0$$

2001 → If $(x^2+y^2+z^2)^2 = a^2x^2+b^2y^2+c^2z^2$ and $lx+my+nz=0$, S.T the maximum values or minimum values of $r^2 = x^2+y^2+z^2$ are given by the equation.

$$\frac{l^2}{r^2-a^2} + \frac{m^2}{r^2-b^2} + \frac{n^2}{r^2-c^2} = 0$$

Sol^n Let $f = x^2+y^2+z^2 = r^2$

$\phi_1 = a^2x^2+b^2y^2+c^2z^2-f^2$

$\phi_2 = lx+my+nz = 0$

Let us consider a function F of independent variables x, y, z

$$\therefore F = (x^2+y^2+z^2) + \lambda_1 (a^2x^2+b^2y^2+c^2z^2-f^2) + 2\lambda_2 (lx+my+nz)$$

~~At Stati~~

$$dF = (2x+2a^2x\lambda_1+2l\lambda_2)dx + (2y+2b^2y\lambda_1+2m\lambda_2)dy + (2z+2c^2z\lambda_1+2\lambda_2 n)dz$$

At stationary points

$$x + a^2x\lambda_1 + \lambda_2 l = 0$$

$$y + b^2y\lambda_1 + \lambda_2 m = 0$$

$$z + c^2z\lambda_1 + \lambda_2 n = 0$$

Multiplying by x, y, z respectively and adding we get

$$\lambda_1(a^2x^2+b^2y^2+c^2z^2)+\lambda_2(lx+my+nz)+x^2+y^2+z^2=0$$

$$\lambda_1(r^2)^2+\lambda_2(0)+r^2=0$$

$$\Rightarrow \lambda_1 r^2+1=0$$

$$\Rightarrow \boxed{\lambda_1=-\frac{1}{r^2}}$$

So $x=\frac{\lambda_2 l r^2}{a^2-r^2}$, $y=\frac{\lambda_2 m r^2}{b^2-r^2}$, $z=\frac{\lambda_2 n r^2}{c^2-r^2}$

Substituting these values in $lx+my+nz=0$

we get

$$\frac{l^2}{a^2-r^2}+\frac{m^2}{b^2-r^2}+\frac{n^2}{c^2-r^2}=0 \quad \text{since } \lambda_2\neq 0.$$

2007, 2017 → Find the rectangular parallelopiped of greatest volume for a given total surface area S, using the Lagrange's method of Multipliers.

2008 → Determine the maximum and minimum distances of the origin from the curve given by the equation $3x^2+4xy+6y^2=140$.

→ Find the maxima and minima of $x^2+y^2+z^2$ subject to the conditions $ax^2+by^2+cz^2=1$ and $lx+my+nz=0$.

Sol ∴ Given that

$f(x,y,z) = x^2+y^2+z^2$ —— ①.

subject to the conditions $ax^2+by^2+cz^2=1$ —— ②

and $lx+my+nz=0$ —— ③.

Let us consider a function F of independent variables x, y, z.

where

$F = x^2+y^2+z^2 + \lambda_1(ax^2+by^2+cz^2-1) + \lambda_2(lx+my+nz)$ —— ④.

$\therefore dF = (2x+2ax\lambda_1+l\lambda_2)dx + (2y+2by\lambda_1+m\lambda_2)dy + (2z+2\lambda_1 cz+n\lambda_2)dz$ —— ⑤. $(\because dF = F_x dx + F_y dy + F_z dz)$

At stationary points, $dF=0$.

$\therefore F_x=0 \Rightarrow 2x+2ax\lambda_1+l\lambda_2=0$

$F_y=0 \Rightarrow 2y+2by\lambda_1+m\lambda_2=0$ —— ⑥.

$F_z=0 \Rightarrow 2z+2\lambda_1 cz+n\lambda_2=0$.

multiplying ⑥ by x, y, z respectively and adding, we get

$2(x^2+y^2+z^2) + 2(ax^2+by^2+cz^2)\lambda_1 + (lx+my+nz)\lambda_2 = 0$

$\Rightarrow 2u + 2(1)\lambda_1 + (0)\lambda_2 = 0$. where $u = x^2+y^2+z^2$

$\Rightarrow \boxed{\lambda_1 = -u}$

from ⑥, we have

$2x+2ax(-u)+l\lambda_2=0 \Rightarrow x = \dfrac{-l\lambda_2}{2(1-au)}$

$2y+2by(-u)+m\lambda_2=0 \Rightarrow y = \dfrac{-m\lambda_2}{2(1-bu)}$

$2z+2cz(-u)+n\lambda_2=0 \Rightarrow z = \dfrac{-n\lambda_2}{2(1-cu)}$

$$(3) \equiv l\left(\frac{-l\lambda_2}{2(1-au)}\right) + m\left(\frac{-m\lambda_2}{2(1-bu)}\right) + n\left(\frac{-n\lambda_2}{2(1-cu)}\right) = 0$$

$$\Rightarrow \lambda_2\left[\frac{l^2}{1-au} + \frac{m^2}{1-bu} + \frac{n^2}{1-cu}\right] = 0. \quad \text{--- (7)}$$

If $\lambda_2 = 0$ then we get $x=y=z=0$

but $(x,y,z) = (0,0,0)$ doesnot satisfy one of the condition of the constraint (1).

So that $\lambda_2 \neq 0$.

from (7), we have

$$\frac{l^2}{1-au} + \frac{m^2}{1-bu} + \frac{n^2}{1-cu} = 0.$$

which gives the maxima and minima of u i.e $u = x^2+y^2+z^2$.

→ Show that the maximum and minimum of radii vectors of the section of the surface $(x^2+y^2+z^2)^2 = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$; by the plane $\lambda x + \mu y + \nu z = 0$ are given by the equation

$$\frac{a^2\lambda^2}{1-a^2r^2} + \frac{b^2\mu^2}{1-b^2r^2} + \frac{c^2\nu^2}{1-c^2r^2} = 0.$$

Sol we have to find the maximum and minimum of the radius vector r, where $r^2 = x^2+y^2+z^2$.

Let it be $f(x,y,z)$

i.e $f(x,y,z) = x^2+y^2+z^2$ --- (1)

subject to the conditions

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = r^4 \quad \text{--- (2)}$$

and $\lambda x + \mu y + \nu z = 0$ --- (3).

Let us consider a function F of the independent variables x, y, z.

where

$F(x,y,z) = (x^2+y^2+z^2) + \lambda_1\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-r^4\right)$
$+ \lambda_2(\lambda x + \mu y + \nu z)$ — (4)

$\therefore dF = \left(2x + \frac{2x}{a^2}\lambda_1 + \lambda\lambda_2\right)dx + \left(2y + \frac{2y}{b^2}\lambda_1 + \mu\lambda_2\right)dy$
$+ \left(2z + \frac{2z}{c^2}\lambda_1 + \nu\lambda_2\right)dz$ — (5) $(\because dF = F_x dx + \ldots + F_z \ldots)$

At stationary points, $dF = 0$

$$\left.\begin{aligned} \therefore F_x = 0 &\Rightarrow 2x + \frac{2x}{a^2}\lambda_1 + \lambda\lambda_2 = 0 \\ F_y = 0 &\Rightarrow 2y + \frac{2y}{b^2}\lambda_1 + \mu\lambda_2 = 0 \\ F_z = 0 &\Rightarrow 2z + \frac{2z}{c^2}\lambda_1 + \nu\lambda_2 = 0 \end{aligned}\right\} \text{— (6)}$$

Multiplying eqn (6) by x, y, z respectively and adding, we get

$2(x^2+y^2+z^2) + 2\lambda_1\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right) + \lambda_2(\lambda x + \mu y + \nu z) = 0$

$2r^2 + 2\lambda_1(r^4) + \lambda_2(0) = 0$

$\Rightarrow r^2 + \lambda_1 r^4 = 0$

$\Rightarrow r^2(1+\lambda_1 r^2) = 0$

$r \neq 0$, $1 + \lambda_1 r^2 = 0$

$\Rightarrow \boxed{\lambda_1 = -\frac{1}{r^2}}$

From (6), we have

$2x + \frac{2x}{a^2}\left(-\frac{1}{r^2}\right) + \lambda\lambda_2 = 0 \Rightarrow 2x\left(1 - \frac{1}{a^2r^2}\right) = -\lambda\lambda_2$

$\Rightarrow x = -\frac{\lambda\lambda_2 a^2 r^2}{(a^2r^2-1)}$

$\Rightarrow x = \frac{a^2r^2\lambda\lambda_2}{1-a^2r^2}$

Similarly, $y = \frac{b^2r^2\mu\lambda_2}{1-b^2r^2}$

and $z = \frac{c^2r^2\nu\lambda_2}{1-c^2r^2}$

Substituting the values of x, y and z in eqn (3).

we get

$$\lambda\left(\frac{a^2 r^2 \lambda\lambda_2}{1-a^2r^2}\right)+\mu\left(\frac{b^2r^2\mu\lambda_2}{1-b^2r^2}\right)+\nu\left(\frac{c^2r^2\nu\lambda_2}{1-c^2r^2}\right)=0$$

$$\Rightarrow \lambda_2 r^2\left(\frac{\lambda^2a^2}{1-a^2r^2}+\frac{\mu^2b^2}{1-b^2r^2}+\frac{\nu^2c^2}{1-c^2r^2}\right)=0$$

$$\Rightarrow \lambda_2\left[\frac{\lambda^2a^2}{1-a^2r^2}+\frac{\mu^2b^2}{1-b^2r^2}+\frac{\nu^2c^2}{1-c^2r^2}\right]=0, \text{ if } r^2\neq 0 \quad (7)$$

If $\lambda_2=0$, then we get $x=y=z=0$.
but $(x,y,z)=(0,0,0)$ doesnot satisfy the other condition of the constraint.

$\therefore \lambda_2 \neq 0$

from (7), we have

$$\frac{\lambda^2a^2}{1-a^2r^2}+\frac{\mu^2b^2}{1-b^2r^2}+\frac{\nu^2c^2}{1-c^2r^2}=0$$

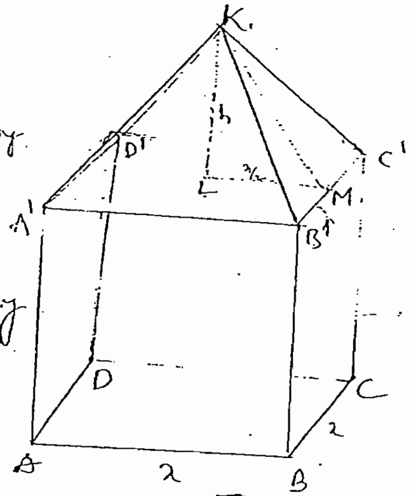
which gives the minima and maxima of $x^2+y^2+z^2$.

HW: find the maximum and minimum values of $\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}$, when $lx+my+nz=0$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

7) A tent on a square base of side x, has its sides vertical of height 'y' and the top is a regular pyramid of height h, find x & y interms of h, if the canvas required for its construction is to be minimum for the tent to have a given capacity.

Sol:

Let 'V' be the volume enclosed by the tent and 'S' be its surface area.



Then V = cuboid $(ABCD, A'B'C'D')$ + pyramid $(K, A'B'C'D')$

$= x^2 y + \frac{1}{3}x^2 h$

$= x^2\left(y + \frac{h}{3}\right)$.

$S = 4\,(ABGF) + 4\,\Delta KGH = 4xy + 4\left(\frac{1}{2} x KM\right)$

$= 4xy + x\sqrt{x^2+4h^2}$ $\left(\because \sqrt{KL^2+LM^2} = \sqrt{h^2+(x/2)^2} = \frac{\sqrt{4h^2+x^2}}{2}\right)$

For constant V, we have

$\delta V = 2x\left(y+\frac{h}{3}\right)(\delta x) + x^2(\delta y) + \frac{x^2}{3}(\delta h) = 0$ $(\delta V = 0)$

For minimum S, we have $(\delta S = 0)$

$\delta S = \left[4y + \sqrt{(x^2+4h^2)} + x\cdot\frac{1}{2}(x^2+4h^2)^{-1/2}\cdot 2x\right]\delta x$
$+ 4x\,\delta y + x\cdot\frac{1}{2}(x^2+4h^2)^{-1/2}\cdot 8h\,\delta h = 0$

By Lagrange's method of Multipliers $\left(\because F = S + \lambda V\right)$

$dF = \left[4y + \sqrt{x^2+4h^2} + x^2(x^2+4h^2)^{-1/2} + \lambda\left(2x\left(y+\frac{h}{3}\right)\right)\right]\delta x$
$+ (4x + \lambda x^2)\,\delta y + \left(4hx(x^2+4h^2)^{-1/2} + \lambda\frac{x^2}{3}\right)\delta h = 0$

At stationary points $dF = 0$.

[illegible]

F_y [illegible]

F_z [illegible]

[illegible]

$\Rightarrow \lambda(4 + \lambda x) = 0$

$\Rightarrow \boxed{\lambda = -\frac{4}{x}}$

Then (3) becomes

$4hx(x^2+4h^2)^{-1/2} - \frac{4y}{x} = 0$

$\Rightarrow \boxed{x = \sqrt{5}\,h}$

now putting $x = \sqrt{5}h$, $\lambda = -\frac{4}{x}$, in (1),

we get

$4y + 3h + \frac{5}{3}h - \frac{4}{x} 2x\left(y + \frac{h}{3}\right) = 0$

$\Rightarrow 4y + \frac{14}{3}h - 8y - \frac{8h}{3} = 0$

$\Rightarrow \boxed{y = h/2}$

→ If the variables x, y, z satisfy the condition $\phi(x)\phi(y)\phi(z) = k^3$ and $\phi(a) = k \neq 0$, $\phi'(a) \neq 0$,

Show that the function $f(x)+f(y)+f(z)$ has a maximum when $x = y = z = a$, provided that $f'(a)\left\{\frac{\phi''(a)}{\phi'(a)} - \frac{\phi'(a)}{\phi(a)}\right\} > f''(a)$

sol^n: Let $F = f(x)+f(y)+f(z)+\lambda\left(\phi(x)\,\phi(y)\,\phi(z) - k^3\right)$

$\therefore dF = F_x\,dx + F_y\,dy + F_z\,dz$.

$\Rightarrow dF = \sum\{f'(x) + \lambda\phi'(x)\,\phi(y)\,\phi(z)\}\,dx$

At Stationary points

$f'(x) + \lambda\phi'(x)\,\phi(y)\,\phi(z) = 0$

$f'(y) + \lambda\,\phi(x)\,\phi'(y)\,\phi(z) = 0$

$f'(z) + \lambda\,\phi(x)\,\phi(y)\,\phi'(z) = 0$

Since the function has a maximum at (a, a, a):

Therefore $f'(a) + \lambda\phi'(a)\,\phi(a)\,\phi(a) = 0$

$$\Rightarrow \lambda = -\frac{f'(a)}{\phi'(a)\,[\phi(a)]^2}$$

$$= -\frac{f'(a)}{k^2\phi'(a)} \quad \left(\because \phi'(a) \neq 0,\ \phi(a) = k\right) \quad \text{—(1)}$$

Now $d^2F = \sum F_{xx}\,dx^2 + 2\sum F_{xy}\,dx\,dy$

$\Rightarrow d^2F = \sum\{f''(x) + \lambda\phi''(x)\,\phi(y)\,\phi(z)\}\,dx^2 + 2\lambda\sum\phi'(x)\,\phi'(y)\,\phi(z)\,dx\,dy.$

At the stationary point (a, a, a)

$$d^2F = \{f''(a) + \lambda K^2 \phi''(a)\} \sum dx^2 + 2\lambda K [\phi'(a)]^2 \sum dx\,dy \quad \text{--- (2)}$$

$$(\because \phi(a) = K)$$

Now $\phi(x)\phi(y)\phi(z) = K^3$

$$\Rightarrow \sum \phi'(x)\phi(y)\phi(z)\,dx = 0$$

$$\Rightarrow K^2\phi'(a)(dx+dy+dz) = 0, \quad \text{at } (a,a,a)$$

$$\Rightarrow dx+dy+dz = 0 \quad (\because K \neq 0,\ \phi'(a) \neq 0)$$

$$\Rightarrow (dx+dy+dz)^2 = 0$$

$$\Rightarrow 2\sum dx\,dy = -\sum dx^2 \quad \text{--- (3)}$$

from (2) and (3), we obtain

$$d^2F = \{f''(a) + \lambda K^2 \phi''(a)\} \sum dx^2 - \lambda K [\phi'(a)]^2 \sum dx^2$$

$$= \left[f''(a) - f'(a)\frac{\phi''(a)}{\phi'(a)} + f'(a)\frac{\phi'(a)}{K}\right] \sum dx^2$$

(by using (1))

$$= f''(a) - f'(a)\left[\frac{\phi''(a)}{\phi'(a)} - \frac{\phi'(a)}{\phi(a)}\right] \quad (\because \phi(a) = K)$$

For a maximum value at (a, a, a), we have $d^2F < 0$

$$\Rightarrow f''(a) - f'(a)\left\{\frac{\phi''(a)}{\phi'(a)} - \frac{\phi'(a)}{\phi(a)}\right\} < 0$$

$$\Rightarrow f''(a) < f'(a)\left\{\frac{\phi''(a)}{\phi'(a)} - \frac{\phi'(a)}{\phi(a)}\right\}$$

$\Rightarrow$ Hence

$$f'(a)\left\{\frac{\phi''(a)}{\phi'(a)} - \frac{\phi'(a)}{\phi(a)}\right\} > f''(a)$$

→ If $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$, show that the stationary value of $a^3x^2+b^3y^2+c^3z^2$ is given by

$$x=\frac{a+b+c}{a},\ y=\frac{a+b+c}{b},\ z=\frac{a+b+c}{c}.$$

Solⁿ: Let $F=(a^3x^2+b^3y^2+c^3z^2)+\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$.

$\therefore F_x=2a^3x-\lambda\left(\frac{1}{x^2}\right),\ F_y=2b^3y-\lambda\left(\frac{1}{y^2}\right)$

$F_z=2c^3z-\lambda\left(\frac{1}{z^2}\right)$ —— (1)

At stationary value $F_x=F_y=F_z=0$ —— (2)

$\therefore xF_x+yF_y+zF_z=0$ gives that

$2(a^3x^2+b^3y^2+c^3z^2)-\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=0$

$\Rightarrow 2f-\lambda(1)=0$ where $f=a^3x^2+b^3y^2+c^3z^2$

$\Rightarrow 2f=\lambda$

From (1) & (2),

$2a^3x^3=2b^3y^3=2c^3z^3(=\lambda)$ $\quad\left(\because F_x=0 \Rightarrow 2a^3x-\frac{\lambda}{x^2}=0 \Rightarrow \lambda=2a^3x^3\right)$

$\Rightarrow ax=by=cz=(k,\ \text{say})$ —— 3

$\Rightarrow \frac{1}{x}=\frac{a}{k};\ \frac{1}{y}=\frac{b}{k};\ \frac{1}{z}=\frac{c}{k}$ —— (3)

$\Rightarrow 1=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{k}(a+b+c)$

$\Rightarrow \boxed{k=a+b+c}$

putting this value of k in (3), we get

$x=\frac{a+b+c}{a},\ y=\frac{a+b+c}{b},\ z=\frac{a+b+c}{c}$

JACOBIANS:

→ If u and v are functions of two independent variables x and y, then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$ is called the Jacobian of u, v with respect to x, y and is denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ or $J\left(\frac{u,v}{x,y}\right)$.

Similarly the Jacobian of u, v, w w.r.t x, y, z is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}.$$

In general, the Jacobian of $u_1, u_2, u_3, \ldots u_n$ ~~w.r.t $x_1, x_2, \ldots x_n$~~ is

$$\frac{\partial(u_1,u_2,u_3,\ldots u_n)}{\partial(x_1,x_2,\ldots x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}.$$

Note: An important application of Jacobians is in connection with the change of variables in multiple integrals.

* Properties of Jacobians:

→ We give below two of the important properties of Jacobians.

For simplicity, the properties are stated in terms of two variables only, but these are evidently true in general.

(I) If $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$

then $JJ' = 1$.

Proof Let $u = f(x,y)$ and $v = g(x,y)$

suppose, on solving for x and y,

we get $x = \phi(u,v)$ and $y = \psi(u,v)$

Then

$$\left.\begin{aligned}
\frac{\partial u}{\partial u} &= 1 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u} \\
\frac{\partial u}{\partial v} &= 0 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v} \\
\frac{\partial v}{\partial u} &= 0 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u} \\
\text{and} \quad \frac{\partial v}{\partial v} &= 1 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v}
\end{aligned}\right\} \text{——(1)}$$

$$\therefore JJ' = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

(after changing the rows and columns of the second determinant)

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} \end{vmatrix}$$

(row by row multiplication)

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad (\because \text{by } (1))$$

$$= 1$$

$$\therefore JJ' = 1$$

(i) **Note :-**

If $J = \dfrac{\partial(u_1, u_2 \ldots u_n)}{\partial(x_1, x_2, \ldots x_n)}$ and $J' = \dfrac{\partial(x_1, x_2 \ldots x_n)}{\partial(u_1, u_2 \ldots u_n)}$

~~then~~ $JJ' = 1$.

(II) **Chain rule for Jacobian** (Jacobian of function of function)

If u, v are functions of r, s and r, s are functions of x, y then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$$

Proof

$$\frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \times \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \times \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{vmatrix}$$

(Interchanging rows and columns of 2nd determinant)

$$= \begin{vmatrix} \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} \\ \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\cdot\frac{\partial s}{\partial y} \end{vmatrix}$$

(row by row multiplication)

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \frac{\partial(u,v)}{\partial(x,y)}$$

Note: If $u_1, u_2, \ldots u_n$ are functions of $y_1, y_2, \ldots y_n$ and $y_1, y_2, \ldots y_n$ are functions of $x_1, x_2, \ldots x_n$

then

$$\frac{\partial(u_1, u_2, \ldots u_n)}{\partial(x_1, x_2, \ldots x_n)} = \frac{\partial(u_1, u_2, \ldots u_n)}{\partial(y_1, y_2, \ldots y_n)} \cdot \frac{\partial(y_1, y_2, \ldots y_n)}{\partial(x_1, x_2, \ldots x_n)}$$

* particular case of Jacobian :

If the functions $u_1, u_2, \ldots u_n$ of $x_1, x_2, \ldots x_n$ are of the form
$u_1 = f_1(x_1)$; $u_2 = f_2(x_1, x_2)$, $u_3 = f_3(x_1, x_2, x_3)$, ……
…… $u_n = f_n(x_1, x_2 \ldots x_n)$

Then it clearly seen that

$$\frac{\partial u_1}{\partial x_2} = 0 = \frac{\partial u_1}{\partial x_3} = \frac{\partial u_1}{\partial x_4} = \cdots = \frac{\partial u_1}{\partial x_n}$$

and $\frac{\partial u_2}{\partial x_3} = 0 = \frac{\partial u_2}{\partial x_4} = \frac{\partial u_2}{\partial x_5} = \cdots = \frac{\partial u_2}{\partial x_n}$ etc.

and therefore

$$\frac{\partial(u_1, u_2, \ldots u_n)}{\partial(x_1, x_2 \ldots x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & 0 & 0 & \cdots & 0 \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & 0 & \cdots & 0 \\ \vdots \\ \frac{\partial u_{n-1}}{\partial x_1} & \frac{\partial u_{n-1}}{\partial x_2} & \cdots & \cdots & 0 \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

$$= \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \cdot \frac{\partial u_3}{\partial x_3} \cdots \frac{\partial u_{n-1}}{\partial x_{n-1}} \cdot \frac{\partial u_n}{\partial x_n}$$

i.e the Jacobian reduces to its leading terms.

problems:

→ If $x = r\cos\theta$, $y = r\sin\theta$, then find $\frac{\partial(x,y)}{\partial(r,\theta)}$ and $\frac{\partial(r,\theta)}{\partial(x,y)}$.

Also prove that $\frac{\partial(r,\theta)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)} = 1$.

Sol: $\dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$

$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$= r$

$\dfrac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial(r)}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$

$= \begin{vmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{vmatrix}$

$\left(\because r = \sqrt{x^2+y^2},\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta,\ \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta \text{ and } \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r},\ \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}\right)$

$= \frac{1}{r}(\cos^2\theta + \sin^2\theta)$

$= \frac{1}{r}(1).$

$= \frac{1}{r}(1).$

Also $\dfrac{\partial(r,\theta)}{\partial(x,y)} \cdot \dfrac{\partial(x,y)}{\partial(r,\theta)} = \frac{1}{r}\cdot r$

$= 1.$

→ If $x = r\cos\theta\cos\phi$, $y = r\cos\theta\sin\phi$, $Z = r\sin\theta$, then ~~find~~ Jacobian of x, y and Z w.r.t r, θ and ϕ

i.e $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$.

→ If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $Z = r\cos\theta$ then show that $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$.

→ If $x = u(1-v)$, $y = uv$, prove that $JJ' = 1$.

Sol since $J = \dfrac{\partial(x,y)}{\partial(u,v)}$, $J' = \dfrac{\partial(u,v)}{\partial(x,y)}$

Now $\frac{\partial x}{\partial u} = 1-v$ | $\frac{\partial y}{\partial u} = v$

$\frac{\partial x}{\partial v} = -u$. | $\frac{\partial y}{\partial v} = u$.

Since $x = u(1-v)$ —① $y = uv$ —②

① ≡ $x = u - uv$

$\Rightarrow x = u - y$ (from ②)

$\Rightarrow \boxed{u = x+y}$

② ≡ $y = (x+y)v$

$\Rightarrow \boxed{v = \frac{y}{x+y}}$

Now $\frac{\partial u}{\partial x} = 1$ | $\frac{\partial v}{\partial x} = \frac{-y}{(x+y)^2} = -\frac{v}{u}$

$\frac{\partial u}{\partial y} = 1$ | $\frac{\partial v}{\partial y} = \frac{(x+y)(1) - y}{(x+y)^2}$

$= \frac{x+y}{(x+y)^2} + \frac{-y}{(x+y)^2} = \frac{1}{u} - \frac{v}{u}$

$= \frac{1-v}{u}$.

Now $JJ' = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \times \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$= \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} \times \begin{vmatrix} 1 & -\frac{v}{u} \\ 1 & \frac{1-v}{u} \end{vmatrix}$

$= (u(1-v) + uv) \times \left(\frac{1-v}{u} + \frac{v}{u}\right)$

$= (u)\left(\frac{1}{u}\right)$

$= \underline{\underline{1}}$.

→ If $x = a\cosh\theta . \cos\phi$, $y = a\sinh\theta \sin\phi$ then show that $\frac{\partial(x,y)}{\partial(\theta,\phi)} = \frac{1}{2}a^2(\cosh 2\theta - \cos 2\phi)$.

Jacobian of Implicit functions:

If y_1, y_2 and x_1, x_2 are implicitly connected by two equations as:

$$f_1(y_1, y_2, x_1, x_2) = 0$$
$$f_2(y_1, y_2, x_1, x_2) = 0$$

then $\dfrac{\partial(f_1, f_2)}{\partial(y_1, y_2)} \cdot \dfrac{\partial(y_1, y_2)}{\partial(x_1, x_2)} = (-1)^2 \dfrac{\partial(f_1, f_2)}{\partial(x_1, x_2)}$

Sol: Given that

$$f_1(y_1, y_2, x_1, x_2) = 0$$
$$f_2(y_1, y_2, x_1, x_2) = 0$$

Now differentiating the relations w.r.t x_1 & x_2

we get

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_1}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial f_1}{\partial y_2}\frac{\partial y_2}{\partial x_1} = 0$$

$$\Rightarrow \frac{\partial f_1}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial f_1}{\partial y_2}\frac{\partial y_2}{\partial x_1} = -\frac{\partial f_1}{\partial x_1} \quad \text{---} (1)$$

Similarly, $\dfrac{\partial f_1}{\partial y_1}\dfrac{\partial y_1}{\partial x_2} + \dfrac{\partial f_1}{\partial y_2}\dfrac{\partial y_2}{\partial x_2} = -\dfrac{\partial f_1}{\partial x_2}$ --- (2)

$$\frac{\partial f_2}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial f_2}{\partial y_2}\frac{\partial y_2}{\partial x_1} = -\frac{\partial f_2}{\partial x_1} \quad \text{---} (3)$$

and $\dfrac{\partial f_2}{\partial y_1}\dfrac{\partial y_1}{\partial x_2} + \dfrac{\partial f_2}{\partial y_2}\dfrac{\partial y_2}{\partial x_2} = -\dfrac{\partial f_2}{\partial x_2}$ --- (4)

Now

L.H.S

$$\frac{\partial(f_1, f_2)}{\partial(y_1, y_2)} \cdot \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} = \begin{vmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{vmatrix} \times \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{vmatrix} \times \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} \end{vmatrix}$$

(Interchanging the rows & columns of 2nd determinant)

$$= \begin{vmatrix} \frac{\partial f_1}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial f_1}{\partial y_2}\frac{\partial y_2}{\partial x_1} & \frac{\partial f_1}{\partial y_1}\frac{\partial y_1}{\partial x_2} + \frac{\partial f_1}{\partial y_2}\frac{\partial y_2}{\partial x_2} \\ \frac{\partial f_2}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial f_2}{\partial y_2}\frac{\partial y_2}{\partial x_1} & \frac{\partial f_2}{\partial y_1}\frac{\partial y_1}{\partial x_2} + \frac{\partial f_2}{\partial y_2}\frac{\partial y_2}{\partial x_2} \end{vmatrix}$$

(row by row multiplication)

$$= \begin{vmatrix} -\frac{\partial f_1}{\partial x_1} & -\frac{\partial f_1}{\partial x_2} \\ -\frac{\partial f_2}{\partial x_1} & -\frac{\partial f_2}{\partial x_2} \end{vmatrix}$$

$$= (-)^2 \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{vmatrix}$$

$$= (-1)^2 \frac{\partial(f_1, f_2)}{\partial(x_1, x_2)} = R.H.S.$$

Note: If $y_1, y_2, \ldots\ldots y_n$ and $x_1, x_2, \ldots\ldots x_n$ are implicitly connected by 'n' equations as

$$f_1(y_1, y_2, \ldots y_n, x_1, x_2, \ldots x_n) = 0$$
$$f_2(y_1, y_2, \ldots y_n, x_1, x_2, \ldots x_n) = 0$$
$$\ldots\ldots\ldots$$
$$f_n(y_1, y_2, \ldots y_n, x_1, x_2, \ldots x_n) = 0$$

then

$$\frac{\partial(f_1, f_2, \ldots\ldots f_n)}{\partial(y_1, y_2, \ldots y_n)} \cdot \frac{\partial(y_1, y_2, \ldots y_n)}{\partial(x_1, x_2, \ldots x_n)} = (-1)^n \frac{\partial(f_1, f_2, \ldots f_n)}{\partial(x_1, x_2, \ldots x_n)}$$

2002, 2004 → Given the roots of the equation in λ.

$(\lambda-x)^3 + (\lambda-y)^3 + (\lambda-z)^3 = 0$ are u, v, w, Then show that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = -2\frac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}$$

Sol^n: Given that

$$(\lambda-x)^3+(\lambda-y)^3+(\lambda-z)^3=0$$

$$\Rightarrow 3\lambda^3-3\lambda^2(x+y+z)+3\lambda(x^2+y^2+z^2)-(x^3+y^3+z^3)=0.$$

If the roots of the above equation be u, v, w.

Then

$$u+v+w=x+y+z$$

$$uv+vw+wu=x^2+y^2+z^2$$

$$uvw=\frac{x^3+y^3+z^3}{3}$$

($\because \alpha,\beta,\gamma$ are the roots of eqn $ax^3+bx^2+cx+d=0$ then $\alpha+\beta+\gamma=-\frac{b}{a}$, $\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}$, $\alpha\beta\gamma=-\frac{d}{a}$)

Now these relation can be written as

$$f_1=u+v+w-x-y-z=0$$

$$f_2=uv+vw+wu-x^2-y^2-z^2=0$$

$$f_3=uvw-\frac{1}{3}(x^3+y^3+z^3)=0.$$

Since $\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}=(-1)^3\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}\cdot\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$

$$\Rightarrow \frac{\partial(u,v,w)}{\partial(x,y,z)}=(-1)^3\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}\Big/\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} \quad \text{(1)}$$

Now $\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}=\begin{vmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z}\\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z}\end{vmatrix}$

$$=\begin{vmatrix}-1 & -1 & -1\\ -2x & -2y & -2z\\ -x^2 & -y^2 & -z^2\end{vmatrix}$$

$$=(-1)^3(2)\begin{vmatrix}1 & 1 & 1\\ x & y & z\\ x^2 & y^2 & z^2\end{vmatrix}$$

$$=(-1)^3(2)\begin{vmatrix}1 & 0 & 0\\ x & y-x & z-x\\ x^2 & y^2-x^2 & z^2-x^2\end{vmatrix} \quad \begin{matrix}C_2\to C_2-C_1\\ C_3\to C_3-C_1\end{matrix}$$

$$= (-2)(y-x)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^2 & y+x & z+x \end{vmatrix}$$

$$= (-2)(y-x)(z-x)\left[1(z+x)-(y+x)\right]$$

$$= (-2)(y-x)(z-x)(z-y)$$

$$= -2(x-y)(y-z)(z-x) \longrightarrow (2)$$

Let $\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ v+w & w+u & u+v \\ vw & wu & uv \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v+w & u-v & u-w \\ vw & w(u-v) & v(u-w) \end{vmatrix} \quad \begin{matrix} C_2 \sim C_2 - C_1 \\ C_3 \sim C_3 - C_1 \end{matrix}$$

$$= (u-v)(u-w)\begin{vmatrix} 1 & 0 & 0 \\ v+w & 1 & 1 \\ vw & w & v \end{vmatrix}$$

$$= (u-v)(u-w)\,1\,(v-w)$$

$$= -(u-v)(v-w)(w-u) \longrightarrow (3)$$

$\therefore$ Substituting (2) & (3) in (1),

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3\,\frac{(-2)(x-y)(y-z)(z-x)}{-(u-v)(v-w)(w-u)}$$

$$= \frac{-2(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$$

→ If $u^3+v^3 = x+y$, $u^2+v^2 = x^2+y^2$

then prove that $\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{3}\frac{(y-x)}{uv(u-v)}$

→ If $u^3+v^3 = x+y$, $u^2+v^2 = x^3+y^3$, show that

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{2}\frac{(y^2-x^2)}{uv(u-v)}$$

→ If $u^2+v+w = x+y^2+z^2$
$u+v^2+w = x^2+y+z^2$
$u+v+w^3 = x^2+y^2+z$

then prove that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1-4(xy+yz+zx)+16xyz}{2-3(u^2+v^2+w^2)+27u^2v^2w^2}$

→ If $u^3+v^3+w^3 = x+y+z$, $u^2+v^2+w^2 = x^3+y^3+z^3$,
$u+v+w = x^2+y^2+z^2$

then show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{(y-z)(y-x)(x-y)}{(u-v)(v-w)(w-u)}$

→ If $u = \frac{x}{(1-r^2)^{1/2}}$, $v = \frac{y}{(1-r^2)^{1/2}}$, $w = \frac{z}{(1-r^2)^{1/2}}$

where $r^2 = x^2+y^2+z^2$,

then show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1}{(1-r^2)^{5/2}}$

Sol: Given $u = \frac{x}{(1-r^2)^{1/2}}$

$\Rightarrow x = u(1-r^2)^{1/2} \Rightarrow x^2 = u^2(1-r^2)$

$\Rightarrow x^2 = u^2(1-x^2-y^2-z^2)$.

Let $f_1 = x^2-u^2(1-x^2-y^2-z^2) = 0$

Similarly $f_2 = y^2-v^2(1-x^2-y^2-z^2) = 0$

$f_3 = z^2-w^2(1-x^2-y^2-z^2) = 0$.

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→ If α, β, γ are the roots of the equation in t,

such that $\frac{u}{a+t}+\frac{v}{b+t}+\frac{w}{c+t} = 1$

then prove that $\frac{\partial(u,v,w)}{\partial(\alpha,\beta,\gamma)} = -\frac{(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)}{(b-c)(c-a)(a-b)}$

* Particular case of Jacobian of implicit functions:

If the implicit relations are given as

$f_1(x_1, x_2, \ldots\ldots x_n, y_1) = 0$

$f_2(x_2, x_3, \ldots\ldots x_n, y_1, y_2) = 0$

$f_3(x_3, x_4, \ldots\ldots x_n, y_1, y_2, y_3) = 0$

$\ldots\ldots\ldots$

$\ldots\ldots\ldots$

$f_n(x_n, y_1, y_2, y_3, \ldots\ldots y_n) = 0$

then it is easily seen that

$$\frac{\partial(f_1, f_2, \ldots\ldots f_n)}{\partial(y_1, y_2, \ldots\ldots y_n)} = \frac{\partial f_1}{\partial y_1} \cdot \frac{\partial f_2}{\partial y_2} \cdot \ldots\ldots \frac{\partial f_n}{\partial y_n}$$

and $$\frac{\partial(f_1, f_2, f_3, \ldots\ldots f_n)}{\partial(x_1, x_2, x_3, \ldots\ldots x_n)} = \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdot \ldots\ldots \frac{\partial f_n}{\partial x_n}$$

$$\therefore \frac{\partial(y_1, y_2, \ldots\ldots y_n)}{\partial(x_1, x_2, \ldots\ldots x_n)} = (-1)^n \frac{\dfrac{\partial(f_1, f_2, \ldots\ldots f_n)}{\partial(x_1, x_2, \ldots\ldots x_n)}}{\dfrac{\partial(f_1, f_2, \ldots\ldots f_n)}{\partial(y_1, y_2, \ldots\ldots y_n)}}$$

$$= (-1)^n \frac{\dfrac{\partial f_1}{\partial x_1} \cdot \dfrac{\partial f_2}{\partial x_2} \cdot \ldots\ldots \dfrac{\partial f_n}{\partial x_n}}{\dfrac{\partial f_1}{\partial y_1} \cdot \dfrac{\partial f_2}{\partial y_2} \cdot \ldots\ldots \dfrac{\partial f_n}{\partial y_n}}$$

Note: Let $y_1, y_2, \ldots\ldots y_n$ be functions of n independent variables $x_1, x_2, \ldots\ldots x_n$. The necessary and sufficient condition that the functions be connected by a relation $f(y_1, y_2, \ldots\ldots y_n) = 0$ is that the Jacobian $\dfrac{\partial(y_1, y_2, \ldots\ldots y_n)}{\partial(x_1, x_2, \ldots\ldots x_n)}$ vanishes identically.

(or)

Let $y_1, y_2, \ldots\ldots y_n$ be functions of n independent variables $x_1, x_2, \ldots x_n$ are functionally related (i.e. $f(y_1, y_2, \ldots y_n) = 0$) iff $\dfrac{\partial(y_1, y_2, \ldots y_n)}{\partial(x_1, x_2, \ldots x_n)} = 0$

→ P.T the functions $u = x+y-z$, $v = x-y+z$ — ① ②

$w = x^2+y^2+z^2-2yz$ ③ are not independent of one another. Find a relation between them.

Sol^n: we have

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2x & 2(y-z) & 2(z-y) \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ x & y-z & z-y \end{vmatrix}$$

$C_2 \to C_2 - C_1$
$C_3 \to C_3 + C_1$

$$= 2\begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 2 \\ x & y-z-x & z-y+x \end{vmatrix}$$

$$= 2 \cdot 1\left[-2(z-y+x) - 2(y-z-x)\right]$$

$$= -4\,(z-y+x+y-z-x)$$

$$= 0$$

Since $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$.

∴ The given functions are not independent i.e, the functions u, v, w are functionally related.

Now we have to find the relation.

From ① & ②

$$\left.\begin{aligned} u+v &= 2x \\ u-v &= 2(y-z) \end{aligned}\right\} \text{—} (4)$$

from ③ $w = x^2+y^2+z^2-2yz$

$$= x^2 + (y-z)^2$$

$$= \left(\frac{u+v}{2}\right)^2 + \left(\frac{u-v}{2}\right)^2 \quad (\text{by } (4))$$

$$= \frac{1}{4}\left[(u+v)^2 + (u-v)^2\right]$$
$$= \frac{1}{4}\left[2(u^2+v^2)\right]$$
$$w = \frac{u^2+v^2}{2}$$
$$\Rightarrow u^2+v^2 = 2w.$$

which is the required relation between the given functions u, v & w.

→ If $u = x^3 + x^2y + x^2z - z^2(x+y+z)$, $v = x+z$, $w = x^2 - z^2 + xy - zy$. Prove that u, v and w are connected by a functional relation.

→ If $u = y\sqrt{(1-x^2)} + x\sqrt{(1-y^2)}$, $v = \sqrt{(1-x^2)(1-y^2)} - xy$. Prove that u and v are not independent and find the relation between them.

→ If $u = x + 2y + z$, $v = x - 2y + 3z$, $w = 2xy - xz + 4yz - 2z^2$ show that they are not independent and find the relation between them.

→ Prove that the functions $u = x(y+z)$, $v = y(x+z)$, $w = z(x-y)$ are not independent and find the relation between them.

→ If $x = \cos u$, $y = \sin u \cos v$, $z = \sin u \sin v \cos w$ then find $\frac{\partial(x, y, z)}{\partial(u, v, w)}$.

Sol^n: ~~Since~~ $x = f(u)$, $y = f(u, v)$, $z = f(u, v, w)$

$$\because \frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} \cdot \frac{\partial z}{\partial w}$$
$$= (-\sin u)(-\sin u \sin v)(-\sin u \sin v \sin w)$$
$$= -\sin^3 u \sin^2 v \sin w.$$

2005 → If $x+y+z=u$, $y+z=uv$, $z=uvw$ then show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$.

Sol: Since $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$

Given that $x+y+z=u$ — (1), $y+z=uv$ — (2), $z=uvw$ — (3)

from (1) & (2)

$x+uv=u$

$\Rightarrow \boxed{x=u-uv}$

(2) & (3)

$y+uvw=uv$

$\Rightarrow \boxed{y=uv-uvw}$

and $\boxed{z=uvw}$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix} \quad R_2 \to R_2+R_3$$

$$= uv\,[u(1-v)+uv]$$

$$= uv\,[u-uv+uv]$$

$$= u^2 v.$$

→ If $u_1 = x_1+x_2+x_3+x_4$

$u_1u_2 = x_2+x_3+x_4$

$u_1u_2u_3 = x_3+x_4$, $u_1u_2u_3u_4 = x_4$.

show that $\frac{\partial(x_1,x_2,x_3,x_4)}{\partial(u_1,u_2,u_3,u_4)} = u_1^3 u_2^2 u_3$.

→ (i) If $y_{m+1}, y_{m+2}, \ldots y_n$ are constant w.r.t $x_1, x_2, \ldots x_m$ (or) (ii) $y_1, y_2, \ldots y_m$ are constant w.r.t $x_{m+1}, x_{m+2}, \ldots x_n$, then

$$\frac{\partial(y_1, y_2, \ldots, y_m, \ldots y_n)}{\partial(x_1, x_2, \ldots x_m, \ldots x_n)} = \frac{\partial(y_1, y_2, \ldots y_m)}{\partial(x_1, x_2, \ldots x_m)} \cdot \frac{\partial(y_{m+1}, y_{m+2}, \ldots y_n)}{\partial(x_{m+1}, x_{m+2}, \ldots x_n)}$$

(i) Given $y_{m+1}, y_{m+2}, \ldots y_n$ are constant w.r.t to $x_1, x_2, \ldots x_m$

i.e, $\frac{\partial y_r}{\partial x_s} = 0$, where $r = m+1, m+2, \ldots n$ ——— (1)
$s = 1, 2, \ldots m$.

$$\therefore \frac{\partial(y_1, y_2, \ldots y_m, \ldots y_n)}{\partial(x_1, x_2, \ldots x_m, \ldots x_n)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_m} & \frac{\partial y_1}{\partial x_{m+1}} & \frac{\partial y_1}{\partial x_{m+2}} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_m} & \frac{\partial y_2}{\partial x_{m+1}} & \cdots & & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_m} & \frac{\partial y_m}{\partial x_{m+1}} & \cdots & & \frac{\partial y_m}{\partial x_n} \\ \frac{\partial y_{m+1}}{\partial x_1} & \frac{\partial y_{m+1}}{\partial x_2} & \cdots & \frac{\partial y_{m+1}}{\partial x_m} & \frac{\partial y_{m+1}}{\partial x_{m+1}} & \cdots & & \frac{\partial y_{m+1}}{\partial x_n} \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_m} & \frac{\partial y_n}{\partial x_{m+1}} & \cdots & & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_m} & \frac{\partial y_1}{\partial x_{m+1}} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_m} & \frac{\partial y_m}{\partial x_{m+1}} & \cdots & \frac{\partial y_m}{\partial x_n} \\ 0 & 0 & \cdots & 0 & \frac{\partial y_{m+1}}{\partial x_{m+1}} & \cdots & \frac{\partial y_{m+1}}{\partial x_n} \\ 0 & 0 & \cdots & 0 & \frac{\partial y_{m+2}}{\partial x_{m+1}} & \cdots & \frac{\partial y_{m+2}}{\partial x_n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \frac{\partial y_n}{\partial x_{m+1}} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

$$= \frac{\partial(y_1, y_2, \ldots y_m)}{\partial(x_1, x_2, \ldots x_m)} \cdot \frac{\partial(y_{m+1}, y_{m+2}, \ldots y_n)}{\partial(x_{m+1}, x_{m+2}, \ldots x_n)}$$

(ii) may also be proved similarly.

Multiple Integrals: The process of integration for one variable can be extended to the functions of more than one variable.

The generalisation of definite integrals is known as multiple integrals.

Double Integrals:

A double integral is the counterpart, in two dimensions, of the definite integral of a function of a single variable. Let A be a finite region of the xy-plane, and let $f(x,y)$ be a function of the independent variables x, y defined at every point in A. Divide the region A into n parts, of areas $\delta A_1, \delta A_2, \ldots \delta A_n$.

Let (x_r, y_r) be any point inside the r^{th} elementary area δA_r.

Form the sum

$$f(x_1,y_1)\,\delta A_1 + f(x_2,y_2)\,\delta A_2 + \cdots + f(x_n,y_n)\,\delta A_n \quad (1)$$

$$\text{i.e } \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r \quad (1)$$

Increase the number of subdivisions taking smaller and smaller elementary areas. Then the limit of the sum (1), if it exists, as n tends infinity and the dimension of each subdivision tend to zero, is called the double integral of

$f(x,y)$ over the region A; and is denoted by

$$\iint_A f(x,y)\, dA \quad \text{--- (2)}$$

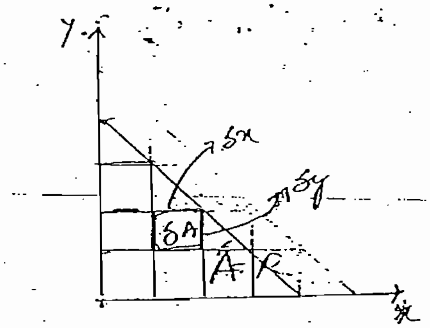
Thus
$$\iint_A f(x,y)\, dA = \lim_{\substack{n\to\infty \\ \delta A \to 0}} \sum_{r=1}^{n} f(x_r, y_r)\, \delta A_r \quad \text{--- (3)}$$

This definition corresponds to the definition

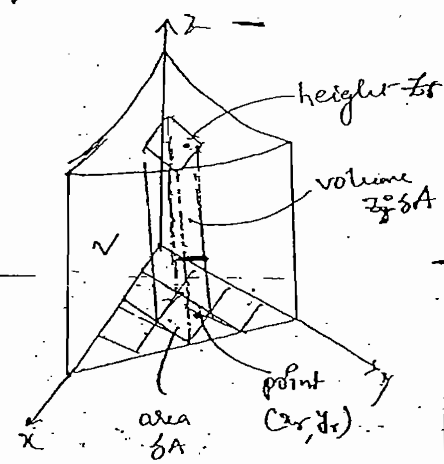
$$\int_a^b f(x)\, dx = \lim_{\substack{n\to\infty \\ \delta x \to 0}} \sum_{r=1}^{n} f(x_r)\, \delta x_r \quad \text{--- (4)}$$

for the definite integral of a single variable.

Just as the definite integral (4) can be interpreted as an area, similarly the double integral (2) can be interpreted as a volume.



Base 'A' cut into small pieces δA.



Solid 'V' cut into thin sticks $\delta V = z\,\delta A$

→ For single integrals, the interval $[a,b]$ is divided into short pieces of length δx.

→ For double integrals, A is divided into small

rectangles of area $\delta A = (\delta x)(\delta y)$.

Above the r^{th} rectangle is a thin stick with small volume. That volume is the base area δA times the height above it – except that this height $z = f(x, y)$ varies from point to point. Therefore we select a point (x_r, y_r) in the r^{th} rectangle and compute the volume from the height above that point.

volume of one stick $= f(x_r, y_r)\, \delta A$

volume of all sticks $= \sum f(x_r, y_r)\, \delta A$.

This is the crucial step for any integral – to see it as a sum of small pieces.

Now take limits $\delta x \to 0$ and $\delta y \to 0$; The height $z = f(x, y)$ is nearly constant over each rectangle (assume that f is continuous function). The sums approaches a limit, which depends only on the base A and the surface above it. The limit is the volume of the solid, and it is the double integral of $f(x, y)$ over A.

$$\iint_A f(x,y)\, dA = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \sum f(x_r, y_r)\, \delta A$$

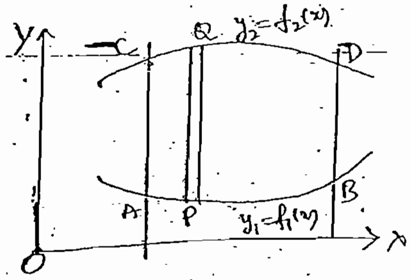
For purposes of evaluation, (2) is expressed as the repeated integral $\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\,dx\,dy$.

→ Its value is found as follows:

(i) When y_1, y_2 are functions of x and x_1, x_2 are constants, $f(x,y)$ is first integrated w.r.t y keeping x fixed between limits y_1, y_2 and then the resulting expression is integrated w.r.t x within the limits x_1, x_2. i.e.,

$$I_1 = \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} f(x,y)\,dy}\,dx}$$

where integration is carried from the inner to outer rectangle; which is geometrically illustrated as shown below.



Here AB and CD are the two curves whose equations are $y_1 = f_1(x)$ and $y_2 = f_2(x)$.

PQ is a vertical strip of width dx.

Then the inner rectangle integral means that the integration is along one edge of

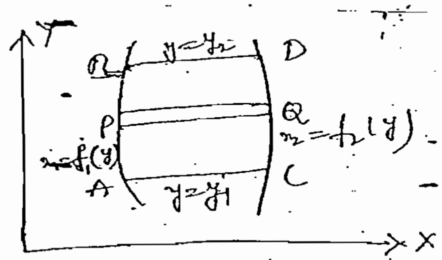
the strip PQ from P to Q (x remaining constant) while the outer rectangle integral corresponds to the sliding of the edge from AC to BD.

Thus the whole region of integration is the area ABDC.

(ii) When x_1, x_2 are functions of y and y_1, y_2 are constants, $f(x,y)$ is first integrated w.r.t x keeping 'y' fixed, with in the limits x_1, x_2 and the resulting expression is integrated w.r.t y between the limits y_1, y_2, i.e.,

$$I_2 = \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x,y)\, dx}\, dy}$$

which is geometrically illustrated as shown below.



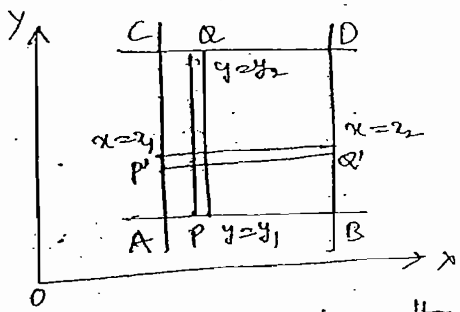
Here AB and CD are the curves $x=f_1(y)$ and $x_2=f_2(y)$. PQ is a horizontal strip of width dy.

Then inner rectangle indicates that the integration is along one edge of this strip from P to Q while the outer rectangle

corresponds to the sliding of this edge from AC to BD.

Thus the whole region of integration is the area ABDC.

(iii) when both pairs of limits are constants, the region of integration is the rectangle ABDC.



In I_1, we integrate along the vertical strip PQ and then slide it from AC to BD.

In I_2, we integrate along the horizontal strip P'Q' and then slide it from AB to CD.

Here obviously $I_1 = I_2$.

Thus for constant limits, it hardly matters whether first integrate w.r.t x and then w.r.t y or vice versa.

→ Evaluate $\int_0^5 \int_0^{x^2} x(x^2+y^2)\,dx\,dy$

Sol^n: Let $I = \int_0^5 \int_0^{x^2} x(x^2+y^2)\,dx\,dy$

$$= \int_0^5 dx \int_0^{x^2} (x^3+y^2x)\,dy$$

$$= \int_0^5 \left[x^3y + \frac{x \cdot y^3}{3}\right]_0^{x^2} dx$$

$$= \int_0^5 x^5 + \frac{x^7}{3}\,dx$$

$$= \left[\frac{x^6}{6} + \frac{x^8}{24}\right]_0^5$$

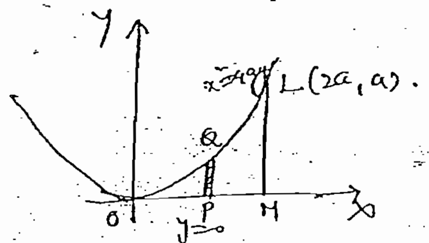
$$= \frac{5^6}{6} + \frac{5^8}{24} = 5^6\left[\frac{1}{6} + \frac{25}{24}\right]$$

$$= \frac{5^6}{24}\left[\frac{29}{24}\right]$$

→ Evaluate $\iint_A xy\,dx\,dy$, where A is the domain bounded by x-axis, ordinate $x=2a$ ~~and~~ the curve $x^2=4ay$.

Sol^n: The line $x=2a$ and the parabola $x^2=4ay$ intersect at $L(2a, a)$.

The figure shows the domain A which is the area OML.



Integrating first over a vertical strip PQ, i.e., w.r.t y from $y=0$ to $y=x^2/4a$ on the parabola and then w.r.t x from $x=0$ to $x=2a$, we have

$$\iint_A xy\,dx\,dy = \int_0^{2a} dx \int_0^{x^2/4a} xy\,dy$$

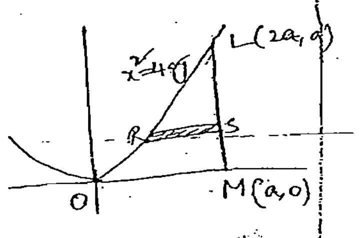
$$= \int_0^{2a} x\left[\frac{y^2}{2}\right]_0^{x^2/4a} dx$$

$$= \frac{1}{32a^2}\int_0^{2a} x^5\,dx$$

$$= \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a}$$

$$= \frac{a^4}{3}.$$

Otherwise integrating first over a horizontal strip RS, i.e., w.r.t x from $x=2\sqrt{ay}$ on the parabola and then w.r.t y from $y=0$ to $y=a$, we get



$$\iint_A xy\,dx\,dy = \int_0^a dy \int_{2\sqrt{ay}}^{2a} xy\,dx$$

$$= \int_0^a y\left[\frac{x^2}{2}\right]_{2\sqrt{ay}}^{2a} dy$$

$$= \int_0^a \frac{(4a^2-4ay)y}{2}\,dy = 2a\int_0^a (ay-y^2)\,dy$$

$$= 2a\left(\frac{ay^2}{2}-\frac{y^3}{3}\right)_0^a$$

$$= \frac{a^4}{3}$$

→ Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2}$

Sol^n:

$$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2} = \int_0^1 dx \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\, dy$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1} \frac{y}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}} dx$$

$\left[\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left(\tan^{-1} 1 - \tan^{-1} 0\right) dx$$

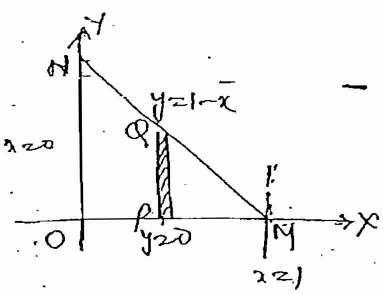
$$= \int_0^1 \left(\frac{\pi}{4} - 0\right) \frac{1}{\sqrt{1+x^2}}\, dx$$

$$= \frac{\pi}{4}\left[\log\left\{x+\sqrt{1+x^2}\right\}\right]_0^1$$

$$= \frac{\pi}{4} \log\left\{1+\sqrt{2}\right\}.$$

→ Evaluate $\iint xy\,dx\,dy$ over the region in the positive quadrant for which $x+y \le 1$.

Sol^n:



The region of integration is the area A bounded by the two axes and the straight line $x+y=1$.

Consider a strip PQ parallel to y-axis.

It has its extremities on $y=0$ and $y=1-x$.

Hence limits of y are from $y=0$ to $y=1-x$.

The limits of x are from $x=0$ to $x=1$.

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Hence the given integral

$$\iint_R xy\,dx\,dy = \int_0^1 \int_0^{1-x} xy\,dx\,dy$$

$$= \int_0^1 x\left(\frac{y^2}{2}\right)_0^{1-x}$$

$$= \frac{1}{2}\int_0^1 x(1-x)^2\,dx$$

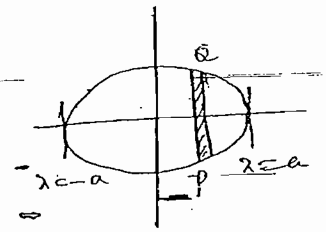
$$= \frac{1}{2}\left(\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right)_0^1$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)$$

$$= \frac{1}{24}.$$

→ Evaluate $\iint (x+y)^2\,dx\,dy$ over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol^n:



For the ellipse

$$\frac{y}{b} = \pm\sqrt{\left(1 - \frac{x^2}{a^2}\right)}$$

$$\Rightarrow y = \pm b\sqrt{1 - \frac{x^2}{a^2}}.$$

Integrating first w.r.t y along a vertical strip PQ which extends from $y = -b\sqrt{1 - \frac{x^2}{a^2}}$ to $y = +b\sqrt{1 - \frac{x^2}{a^2}}$.

To cover the region, we then integrate w.r.t x from $x = a$ to $x = a$.

$\therefore$ The given integral

$$\iint (x+y)^2 = \int_{-a}^{a}\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} (x^2+2xy+y^2)\,dx\,dy$$

$$= \int_{-a}^{a}\left[x^2y+2x\frac{y^2}{2}+\frac{y^3}{3}\right]_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$= \int_{-a}^{a}\left\{2bx^2\sqrt{1-\frac{x^2}{a^2}}+0+\frac{2}{3}b^3\left(\sqrt{1-\frac{x^2}{a^2}}\right)^3\right\}dx$$

$$= 2\int_{-a}^{a}\left\{bx^2\sqrt{1-\frac{x^2}{a^2}}+\frac{b^3}{3}\left(1-\frac{x^2}{a^2}\right)^{3/2}\right\}dx$$

$$= 4b\int_{0}^{a}\left\{x^2\sqrt{1-\frac{x^2}{a^2}}+\frac{b^2}{3}\left(1-\frac{x^2}{a^2}\right)^{3/2}\right\}dx$$

putting $x = a\sin\theta \Rightarrow dx = a\cos\theta\, d\theta$

Limits: ~~when~~ $x=0$; $\theta=0$

$x=a$; $\theta=\pi/2$

$$= 4b\int_{0}^{\pi/2}\left\{a^2\sin^2\theta\cos\theta+\frac{1}{3}b^2\cos^3\theta\right\}a\cos\theta\,d\theta$$

$$= 4ab\int_{0}^{\pi/2}\left\{a^2\sin^2\theta\cos^2\theta+\frac{b^2}{3}\cos^4\theta\right\}d\theta.$$

$$= 4ab\left\{\int_{0}^{\pi/2}\frac{a^2}{4}\sin^2 2\theta\,d\theta+\frac{b^2}{3}\int_{0}^{\pi/2}\cos^4\theta\,d\theta\right\}$$

$$= 4ab\left[\int_{0}^{\pi/2}\frac{a^2}{4}\left(\frac{1-\cos 4\theta}{2}\right)d\theta+\frac{b^2}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$= 4ab\left[\frac{a^2}{8}\left(\theta+\frac{\sin 8\theta}{8}\right)_{0}^{\pi/2}+\frac{b^2\pi}{16}\right]$$

$$= 4ab\left[\frac{a^2}{8}\left(\frac{\pi}{2}+0\right)+\frac{\pi b^2}{16}\right]$$

$$= \frac{1}{4}\pi ab\,(a^2+b^2)$$

Show that $\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3} dy \neq \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3} dx$ (7)

Find the values of the two integrals.

Soln: $LHS = \int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3} dy$

$$= \int_0^1 dx \int_0^1 \frac{2x-(x+y)}{(x+y)^3} dy$$

$$= \int_0^1 dx \int_0^1 \left\{ \frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2} \right\} dy$$

$$= \int_0^1 \left[\frac{-x}{(x+y)^2} + \frac{1}{x+y} \right]_0^1 dx \quad \left(\int \frac{1}{x^n} dx = \frac{x^{-n}}{-n} \right)$$

$$= \int_0^1 \left[-\frac{x}{(1+x)^2} + \frac{1}{x} + \frac{1}{1+x} - \frac{1}{x} \right] dx$$

$$= \int_0^1 \frac{dx}{(1+x)^2}$$

$$= \left[-\frac{1}{1+x} \right]_0^1 = -\frac{1}{2} + 1 = \frac{1}{2}.$$

$RHS = \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3} dx$

$$= \int_0^1 dy \int_0^1 \frac{x+y-2y}{(x+y)^3} dx$$

$$= \int_0^1 dy \int_0^1 \left\{ \frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3} \right\} dx$$

$$= \int_0^1 \left[-\frac{1}{x+y} + \frac{y}{(x+y)^2} \right]_0^1 dy$$

$$= \int_0^1 \left[-\frac{1}{1+y} + \frac{1}{y} + \frac{y}{(1+y)^2} - \frac{1}{y} \right] dy$$

$$= -\int_0^1 \frac{dy}{(1+y)^2} = \left[\frac{1}{1+y} \right]_0^1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$\therefore LHS \neq RHS$

Evaluate the following double integrals:

(1) $\int_0^1 \int_0^2 (x+2)\,dx\,dy$; Ans: 5

(2) $\int_0^a \int_0^b (x^2+y^2)\,dx\,dy$; Ans: $\frac{1}{3}ab(a^2+b^2)$

(3) $\int_1^a \int_1^b \frac{dx\,dy}{xy}$; Ans: $\log a \log b$

(4) $\int_1^2 \int_0^x \frac{dx\,dy}{x^2+y^2}$; Ans: $\frac{1}{4}\log_e 2$

(5) $\int_1^2 \int_0^{x/2} y\,dy\,dx$; Ans: $\frac{7}{6}$

(6) $\int_0^{\pi/2} \int_{\pi/2}^{\pi} \cos(x+y)\,dy\,dx$; Ans: -2

(7) $\int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 y\,dx\,dy$; Ans: $a^5/15$

(8) $\int_0^1 \int_0^{\sqrt{1-y^2}} 4y\,dy\,dx$; Ans: $4/3$

(9) $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy$; Ans: $3/35$

(10) $\int_0^1 \int_0^{x^2} e^{y/x}\,dx\,dy$; Ans: $1/2$

(11) $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$; Ans: $\frac{\pi a^3}{6}$

(12) $\int_0^2 \int_0^{\sqrt{2x-x^2}} x\,dx\,dy$; Ans: $\pi/2$

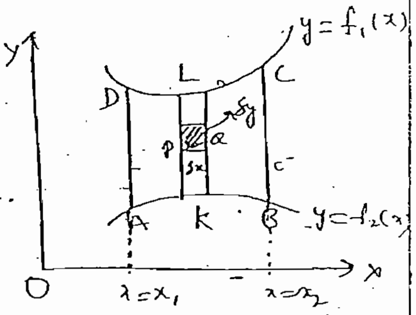
(13) Evaluate $\iint x^2y^2\,dx\,dy$ over the region $x^2+y^2 \le 1$. Ans: $\pi/24$.

(14) Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the region in the positive quadrant for which $x+y \le 1$. Ans: $1/6$.

(15) Evaluate $\iint \frac{xy}{\sqrt{1-y^2}}\,dx\,dy$ over the positive quadrant of the circle $x^2+y^2=1$. Ans: $1/6$.

Area enclosed by plane curves:

Consider the area enclosed by the curves $y = f_1(x)$ and $y = f_2(x)$ and the ordinates $x = x_1$, $x = x_2$ as shown in the figure.



Divide this area into vertical strips of width δx.

If $P(x, y)$, $Q(x+\delta x, y+\delta y)$ be two neighbouring points, then the area of the small rectangle $PQ = \delta x\, \delta y$

$\therefore$ Area of strip $KL = \underset{y \to 0}{Lt} \sum \delta x\, \delta y$.

Since for all rectangles in this strip δx is the same and y varies from $y = f_1(x)$ to $y = f_2(x)$.

$$\therefore \text{Area of the strip } KL = \delta x \underset{\delta y \to 0}{Lt} \sum_{f_1(x)}^{f_2(x)} dy$$

$$= \delta x \int_{f_1(x)}^{f_2(x)} dy$$

Now adding up all such strips from $x = x_1$ to $x = x_2$ we get the area ABCD

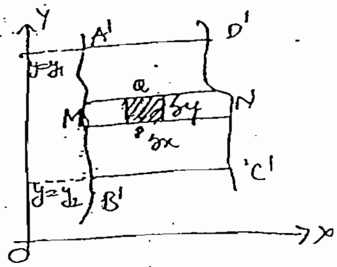
$$= \underset{\delta x \to 0}{Lt} \sum_{x_1}^{x_2} \delta x \int_{f_1(x)}^{f_2(x)} dy$$

$$= \int_{x_1}^{x_2} dx \int_{f_1(x)}^{f_2(x)} dy$$

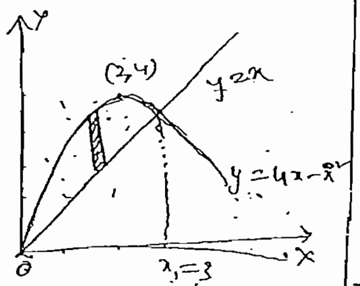
$$= \int_{x_1}^{x_2} \int_{f_1(x)}^{f_2(x)} dx\, dy$$

Similarly dividing the area A'B'C'D' into horizontal strips of width δy, we get the area

$$A'B'C'D' = \int_{y_1}^{y_2}\int_{f_1(y)}^{f_2(y)} dx\,dy$$



→ Find the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.



Sol^n: The equation of the parabola $y = 4x - x^2$ may be written as $(x-2)^2 = -(y-4)$

i.e., this parabola has the vertex at the point $(2,4)$ and its concavity is downwards.

The point of intersection of two curves are given as follows

$$4x - x^2 = x$$
$$\Rightarrow x^2 - 3x = 0$$
$$\Rightarrow x(x-3) = 0$$
$$\Rightarrow x = 0 \text{ (or) } x = 3.$$

and hence from $y = x$, we get

$y = 0$ at $x = 0$

$y = 3$ at $x = 3$

∴ The points of intersection of the two curves are $(0,0)$, $(3,3)$.

∴ The area can be considered as lying between the curves $y=x$, $y=4x-x^2$, $x=0$ and $x=3$

So integrating along a vertical strip first, i.e. y from $y=x$ to $y=4x-x^2$ and then w.r.t x from $x=0$ to $x=3$.

∴ The required area $= \int_0^3 \int_{y=x}^{4x-x^2} dx\,dy$

$$= \int_0^3 [y]_x^{4x-x^2} dx$$

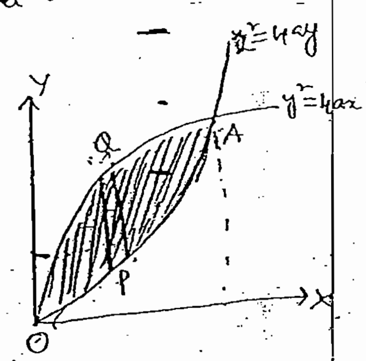
$$= \int_0^3 (4x-x^2-x)dx$$

$$= \int_0^3 (3x-x^2)dx$$

$$= \left[\frac{3x^2}{2}-\frac{x^3}{3}\right]_0^3$$

$$= \frac{27}{2}-9 = \frac{9}{2}$$

→ Show that the area between the parabolas $y^2=4ax$ and $x^2=4ay$ is $\frac{16}{3}a^2$.



Sol^n: Solving the equations $y^2=4ax$ and $x^2=4ay$, the parabolas intersect at $O(0,0)$ and $A(4a,4a)$. As such for the shaded area between these parabolas (as shown in the figure)

∴ x varies from 0 to $4a$ and y varies from $y=x^2/4a$ to $y=2\sqrt{ax}$.

Hence the required area

$$= \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dx\,dy$$

$$= \int_0^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a}\right) dx$$

$$= \left[2\sqrt{a}\ \frac{2}{3} x^{3/2} - \frac{1}{4a}\ \frac{x^3}{3}\right]_0^{4a}$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2$$

→ Find by double integration, the area enclosed by the curves $y = 3x/x^2+2$ and $4y = x^2$.

Ans: $\frac{2}{3}\log_e 3 - \frac{2}{3}$

→ Find by double integration the area of the region enclosed by the following curves:

(1) $x^2+y^2 = a^2$ and $x+y=a$ (in the first quadrant)

Ans: $\frac{(\pi-2)a^2}{4}$

(2) $y^2 = x^3$ and $y = x$. Ans: $\frac{1}{10}$

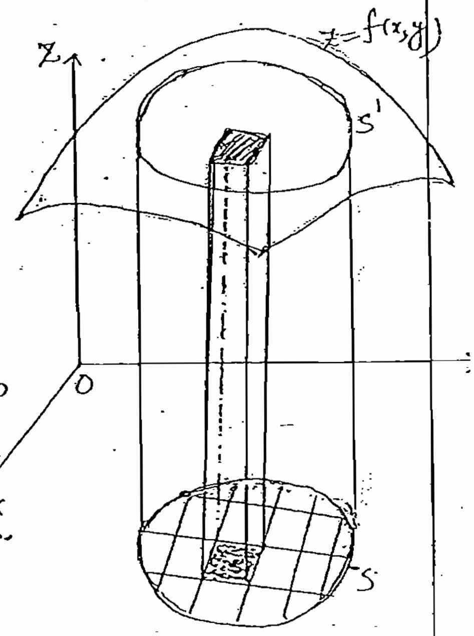
(3) $9xy = 4$ and $2x+y = 2$. Ans: $\frac{1}{3} - \frac{4}{9}\log_e 2$.

(4) $(x^2+4a^2)y = 8a^3$, $2y = x$ and $x = 0$ Ans: $(\pi-1)a^2$.

Volume as double integral:

Consider a surface $z = f(x, y)$.

Let the orthogonal projection on xy-plane of its portion S' be the area S.



Divide S into elementary rectangles of area $\delta x \delta y$ by drawing lines parallel to x and y axes. With each of these rectangles as base, erect a prism having its length parallel to OZ.

$\therefore$ volume of this prism between S and the given surface $z = f(x, y)$ is $z \, \delta x \, \delta y$.

Hence the volume of the solid cylinder on S as base, bounded by the given surface with generators parallel to the z-axis:

$$= \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \sum z \, \delta x \, \delta y$$

$$= \iint z \, dx \, dy$$

$$= \iint f(x, y) \, dx \, dy$$

where the integration is carried over the area S.

i.e, If the region S may be considered as enclosed by the curves $y = f_1(x)$, $y = f_2(x)$,

$x = a$ and $x = b$, we can write volume as:

$$\int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\,dx\,dy$$

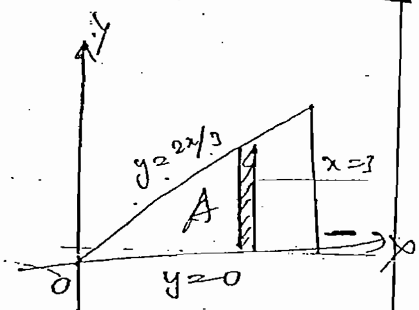
Note: When writing the integral for the volume, the integrand $f(x,y)$ is taken from the surface $z = f(x,y)$ which covers the top of the volume while the limits a, b, f_1, f_2 are taken from the base area S in the xy-plane.

→ Find the volume under the plane $x+y+z=6$ and above triangle in the xy-plane bounded by $2x = 3y$, $y = 0$, $x = 3$.

Sol: The required volume V

$$= \iint_A z\,dA$$

$$= \iint_A (6-x-y)\,dA,$$



Where A is the region shown in the figure.

Integrating along a vertical strip first, we have

$$V = \int_0^3 \int_0^{2x/3} (6-x-y)\,dx\,dy$$

$$= \int_0^3 \left(6y - xy - \frac{y^2}{2}\right)_0^{2x/3} dx$$

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$$= \int_0^3 \left(4x - \frac{2}{3}x^2 - \frac{2}{9}x^2\right) dx$$

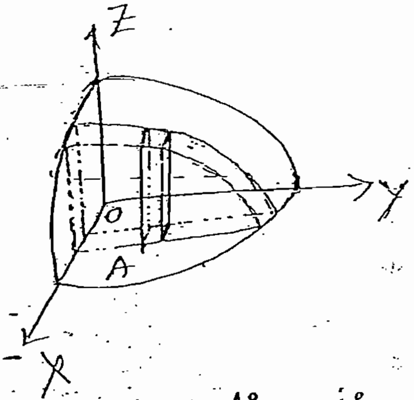
$$= \int_0^3 \left(4x - \frac{8}{9}x^2\right) dx.$$

$$= \left(2x^2 - \frac{8}{27}x^3\right)_0^3$$

$$= 18 - 8 = 10.$$

2005 P-I Q-3 → Find the volume in the positive octant of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Solⁿ: The required volume lies between the ellipsoid $z = c\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}$ and the plane XOY, and is bounded on the sides by the planes $x=0$, $y=0$.



The given ellipsoid cuts XOY plane in the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad z=0.$$

Therefore the region A, above which the required volume lies, is bounded by curves

$$y=0, \; y = b\sqrt{1-x^2/a^2},$$

$$x=0, \text{ and } x=a.$$

Hence, the required volume

$$= \iint_A z\,dA$$

$$= \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}\,dx\,dy$$

$$= c\int_0^a \int_0^Y \sqrt{\frac{Y^2}{b^2}-\frac{y^2}{b^2}}\,dx\,dy \quad \text{on putting } \sqrt{1-\frac{x^2}{a^2}}=\frac{Y}{b}$$

$$= \frac{c}{b}\int_0^a \left[\frac{1}{2}y\sqrt{Y^2-y^2}+\frac{1}{2}Y^2\sin^{-1}\frac{y}{Y}\right]_0^Y dx$$

$$= \frac{c}{b}\int_0^a \frac{1}{2}Y^2\,\frac{\pi}{2}\,dx$$

$$= \frac{\pi c}{4b}\int_0^a Y^2\,dx$$

$$= \frac{\pi c}{4b}\int_0^a b^2\left(1-\frac{x^2}{a^2}\right)dx \quad \left(\because Y = b\sqrt{1-\frac{x^2}{a^2}}\right)$$

$$= \frac{1}{4}\pi bc\left[x-\frac{x^3}{3a^2}\right]_0^a$$

$$= \frac{1}{4}\pi bc\left[a-\frac{a^3}{3a^2}\right]$$

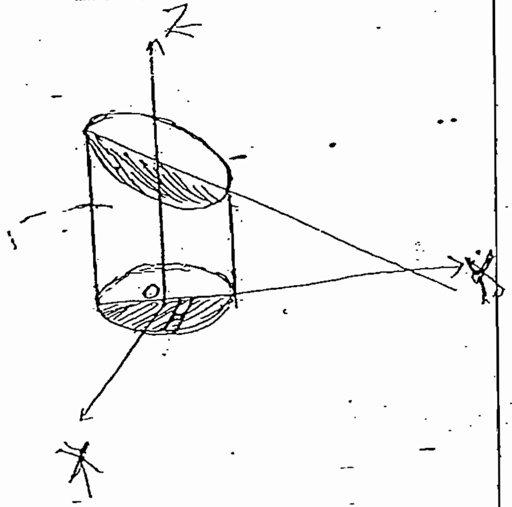
$$= \frac{1}{4}\pi bc\,\frac{2a}{3}$$

$$= \frac{1}{6}\pi abc \quad \text{Ans}$$

→ Find the volume bounded by the cylinder $x^2+y^2=4$ and the plane $y+z=4$ and $z=0$.

Solⁿ: From the figure, it is self-evident that $z=4-y$ is to be integrated over the circle $x^2+y^2=4$ in the xy-plane. To cover the shaded

half of this circle, x varies from 0 to $\sqrt{4-y^2}$ and y varies from -2 to 2.



$\therefore$ Required volume

$$= 2\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} z\,dx\,dy$$

$$= 2\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} (4-y)\,dx\,dy$$

$$= 2\int_{-2}^{2} (4-y)\,(x)_0^{\sqrt{4-y^2}}\,dy$$

$$= 2\int_{-2}^{2} (4-y)\sqrt{4-y^2}\,dy$$

$$= 2\int_{-2}^{2} 4\sqrt{4-y^2}\,dy - 2\int_{-2}^{2} y\sqrt{4-y^2}\,dy$$

$$= 8\int_{-2}^{2}\sqrt{4-y^2}\,dy$$ (Here the second term vanishes as the integrand is an odd function)

$$= 8\left[\frac{y\sqrt{4-y^2}}{2} + \frac{4}{2}\sin^{-1}\frac{y}{2}\right]_{-2}^{2}$$ $\left[\because \int\sqrt{a^2-x^2} = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$

$$= 8\left[0 + 2\sin^{-1}(1) - 0 - 2\sin^{-1}(-1)\right]$$

$$= 8\left[2\cdot\frac{\pi}{2} + 2\cdot\frac{\pi}{2}\right]$$

$$= 16\pi.$$

→ Find the volume of the cylinder $x^2+y^2=ax$ bounded by the planes ~~z=0~~ and $z=x$. Ans: $\pi a^3/8$

→ Find the volume under the plane $x+z=2$, above $z=0$ and with in the cylinder $x^2+y^2=4$. Ans: 8π

→ Find the volume under the plane $z = x+y$ and above the area cut from the first quadrant by ellipse $4x^2+9y^2=36$. Ans: 10

→ Find the volume bounded by the co-ordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. Ans: $\frac{abc}{6}$

→ Find the volume bounded by $4z = 16-x^2-y^2$ and the plane $z=0$. Ans: 16π

→ Find the volume enclosed by the cylinders $y^2=x$ and $x^2+y^2=a^2$ and the plane $z=0$. Ans: $\pi a^4/4$.

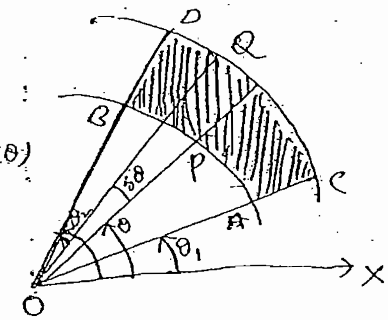
→ Find the volume in the first octant bounded by the parabolic cylinders $z=9-x^2$, $x=3-y^2$. Ans: $\frac{102\sqrt{3}}{35}$

→ Find the volume in the first octant bounded by $z=x^2+y^2$ and $y=1-x^2$. Ans: $2/7$

→ Find the volume inside the paraboloid $x^2+4z^2+8y=16$ and on the positive side of xz-plane

Polar co-ordinates:

To evaluate $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\,dr\,d\theta$, we first integrate w.r.t r between limits $r=r_1$ and $r=r_2$ keeping θ fixed and the resulting expression is integrated w.r.t θ from θ_1 to θ_2. In this integral



Here AB and CD are the curves $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$ bounded by the lines $\theta=\theta_1$ and $\theta=\theta_2$. PQ is a wedge of angular thickness $\delta\theta$.

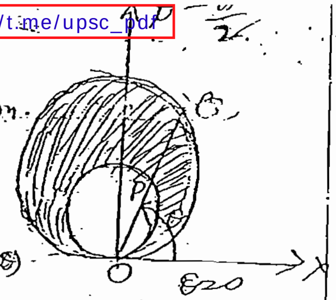
Then $\int_{r_1}^{r_2} f(r,\theta)\,dr$ indicates that the integration is along PQ from P to Q while integration w.r.t θ corresponds to the turning of PQ from AC to BD.

Thus the whole region of integration is the area ACDB. The order of integration may be changed with appropriate changes in the limits.

→ Calculate $\iint r^3\,dr\,d\theta$ over the area included between the circles $r=2\sin\theta$ and $r=4\sin\theta$.

Sol^n: Given circles $r=2\sin\theta$ and $r=4\sin\theta$ are as shown in the figure.

The shaded area between these circles is the region of integration. If we integrate first w.r.t r, then its limits are from $P(r=2\sin\theta)$ to $Q(r=4\sin\theta)$ and to cover the whole region θ varies from 0 to π.



Thus the required integral is

$$I = \int_0^{\pi} d\theta \int_{2\sin\theta}^{4\sin\theta} r^3 dr$$

$$= \int_0^{\pi} d\theta \left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta}$$

$$= \frac{1}{4}\int_0^{\pi} (256-16)\sin^4\theta\, d\theta$$

$$= \frac{240}{4}\int_0^{\pi} \sin^4\theta\, d\theta$$

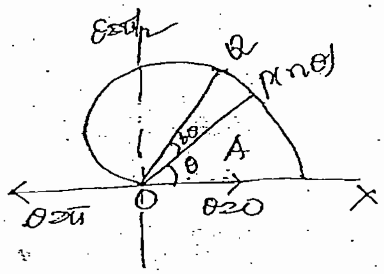
$$= 60 \times 2\int_0^{\pi/2} \sin^4\theta\, d\theta$$

$$= 120 \times \frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}$$

$$= 22.5\pi$$

→ Integrate $r\sin\theta$ over the area of the cardioid $r=a(1+\cos\theta)$ above the initial line.

Sol^n: Here the region of integration A can be covered by



radial strips whose ends are $r=0$ and $r=a(1+\cos\theta)$ i.e.,

The strips start from $\theta=0$ and end at $\theta=\pi$.

Therefore the required integral

$$= \iint_A r\sin\theta \, dA = \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\sin\theta \; r\,d\theta\,dr.$$

$$= \int_0^{\pi} \sin\theta \left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{3}a^3 \int_0^{\pi} \sin\theta\,(1+\cos\theta)^3\,d\theta$$

$$= \frac{1}{3}a^3 \int_0^{\pi} 2\sin\theta/2\,\cos\theta/2\,(2\cos^2\theta/2)^3\,d\theta$$

$$= \frac{16}{3}a^3 \int_0^{\pi} \sin\theta/2\,\cos^7\theta/2\,d\theta$$

putting $\theta=2\phi \Rightarrow d\theta=2d\phi$.

Limits: $\phi=0$ when $\theta=0$

$\phi=\pi/2$ when $\theta=\pi$

$$= \frac{16}{3}a^3 \int_0^{\pi/2} \sin\phi\,\cos^7\phi\; 2d\phi$$

$$= \frac{32}{3}a^3 \left[-\frac{\cos^8\phi}{8}\right]_0^{\pi/2}$$

$$= \frac{4}{3}a^3.$$

→ Evaluate $\iint_A r^2\sin\theta\,d\theta\,dr$ over the area of Cardioid $r=a(1+\cos\theta)$ above the initial line.

Ans: $\frac{4}{3}a^3$.

radial strips whose ends are $r=0$ and $r=a(1+\cos\theta)$ i.e.,

The strips start from $\theta=0$ and end at $\theta=\pi$.

Therefore the required integral

$$= \iint_A r\sin\theta \, dA = \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\sin\theta \; r\,d\theta\,dr.$$

$$= \int_0^{\pi} \sin\theta \left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{3}a^3 \int_0^{\pi} \sin\theta \,(1+\cos\theta)^3 d\theta$$

$$= \frac{1}{3}a^3 \int_0^{\pi} 2\sin\theta/2 \cos\theta/2 \,(2\cos^2\theta/2)^3 d\theta$$

$$= \frac{16}{3}a^3 \int_0^{\pi} \sin\theta/2 \cos^7\theta/2 \, d\theta$$

putting $\theta = 2\phi \Rightarrow d\theta = 2d\phi$.

Limits: $\phi = 0$ when $\theta=0$

$\phi = \pi/2$ when $\theta = \pi$

$$= \frac{16}{3}a^3 \int_0^{\pi/2} \sin\phi \cos^7\phi \; 2d\phi$$

$$= \frac{32}{3}a^3 \left[-\frac{\cos^8\phi}{8}\right]_0^{\pi/2}$$

$$= \frac{4}{3}a^3.$$

→ Evaluate $\iint_A r^2\sin\theta \, d\theta\, dr$ over the area of cardioid $r=a(1+\cos\theta)$ above the initial line.

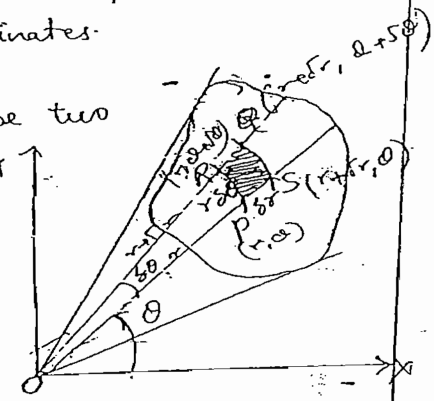
Ans: $\frac{4}{3}a^3$.

Area enclosed by plane curves

polar co-ordinates:

Consider an area A enclosed by a curve whose equation is in polar co-ordinates.

Let $P(r, \theta)$, $Q(r+\delta r, \theta+\delta\theta)$ be two neighbouring points.



Mark circular areas of radii r and $r+\delta r$ meeting OQ in R and OP in S.

Since arc $PR = r\delta\theta$ $\quad (\because l = r\theta)$

and $PS = \delta r$.

$\therefore$ Area of the curvilinear rectangle PRQS is approximately $= PR \cdot PS$

$= r\delta\theta \cdot \delta r$

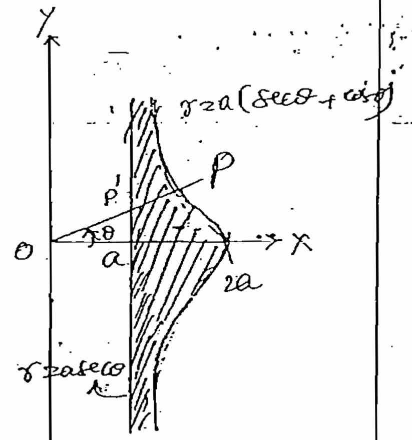
If the whole area is divided into such curvilinear rectangles, the sum $\sum\sum r\delta\theta\delta r$ taken for all these rectangles, gives in the limit the area A.

Hence $A = \lim\limits_{\substack{\delta r \to 0 \\ \delta\theta \to 0}} \sum\sum r\delta\theta\delta r = \iint r\,d\theta\,dr$.

where the limits are to be so chosen as to cover the entire area.

→ Calculate the area included between the curve $r = a(\sec\theta + \cos\theta)$ and its asymptote

Sol^n: The curve is symmetrical about the initial line and has an asymptote $r = a\sec\theta$.



Draw any line OP cutting the curve at P and its asymptote at P'.

Along this line, θ is constant and r varies from $a\sec\theta$ at P' to $a(\sec\theta+\cos\theta)$ at P. Then to get the upper half of the area, θ varies from 0 to $\pi/2$.

$\therefore$ The required area

$$= 2\int_0^{\pi/2}\int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a\sec\theta}^{a(\sec\theta+\cos\theta)} d\theta$$

$$= \frac{2a^2}{2}\int_0^{\pi/2}\left[(\sec\theta+\cos\theta)^2 - \sec^2\theta\right]d\theta$$

$$= a^2\int_0^{\pi/2}(2+\cos^2\theta)\,d\theta = a^2\left[2\theta\right]_0^{\pi/2} + \frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{5\pi a^2}{4} \text{ Ans.}$$

→ Find by double integration, the area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1-\cos\theta)$.

Ans:

Change of order of Integration:

The integral $\iint U\,dx\,dy$ is first integrated with respect to the variable 'y', then limits of y are substituted (which in general may be function of x), and the result is integrated with respect to 'x'. But if we want to change $\iint U\,dx\,dy$ to $\iint U\,dy\,dx$ then we have to find the new limits of x as functions of y.

i.e., in a double integral with variable limits, the change of order of integration changes the limits of integration. While doing so, sometimes it is required to split up the region of integration and the given integral is expressed as the sum of a number of double integrals with changed limits.

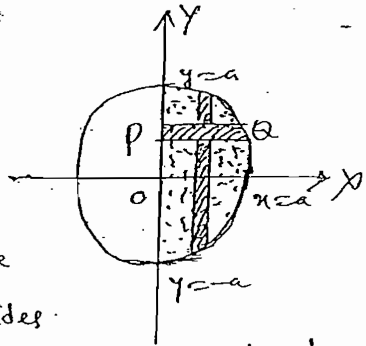
To fix up the new limits, it is always advisable to draw a rough sketch of the region of integration.

The change of order of integration quite often facilitates the evaluation of a double integral.

Change

→ Change the order of integration in the integral $I = \int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$.

Sol^n: Here the elementary strip is parallel to x-axis (such as PQ) and extends from $x=0$ to $x=\sqrt{a^2-y^2}$ (i.e, the circle $x^2+y^2=a^2$) and this strip slides from $y=-a$ to $y=a$. This shaded semi-circular area is, therefore, the region of integration.



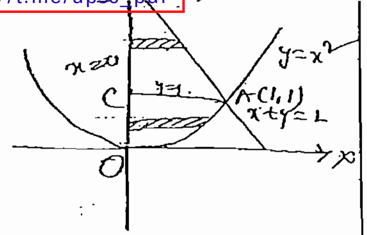
On changing the order of integration, we first integrate w.r.t y along a vertical strip RS which extends from R $[y=-\sqrt{a^2-x^2}]$ to S $[y=\sqrt{a^2-x^2}]$. To cover the given region, we then integrate w.r.t x from $x=a$ to $x=a$.

Thus $I = \int_0^a dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)\,dy$

$$= \int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx.$$

→ Change the order of integration in $I = \int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy$ and hence evaluate the same.

Sol^n:



The given limits show that the region of integration is bounded by the curves

$y=x^2$, $y=2-x$, $x=0$, $x=1$.

The first is a parabola with vertex at the origin and the second the straight line $y=2-x$.

These intersect at the point $(1,1)$.

Therefore the region of integration is OAB.

When we integrate w.r.t x first along a horizontal strip, the strip starts from $x=0$

But some of the strips end on OA while the others end on AB. i.e., At A strips parallel to x-axis change their character.

Hence through the point A, draw a straight line $(y=1)$ CA parallel to the x-axis. This straight line CA divides the region OAB into two parts namely OAC and ABC.

In the region OAC, the strip parallel to x-axis has its extremities on $x=0$ and $y=x^2$.

Hence limits of x are from $x=0$ to $x=\sqrt{y}$

As the point A is $(1,1)$, the limits of y are from $y=0$ to $y=1$

Again in the region ABC, the strip parallel to x-axis has its extremities on $x=0$ and $y=2-x$.
Hence limits of x are from $x=0$ to $x=2-y$.
The limits of y are from $y=1$ to $y=2$.

Hence changing the order of integration the given integral is

$$\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy = \int_0^1 dy\int_0^{\sqrt{y}} xy\,dx + \int_1^2 dy\int_0^{2-y} xy\,dx$$

$$= \int_0^1 \left[\frac{x^2}{2}y\right]_0^{\sqrt{y}} dy + \int_1^2 \left[\frac{x^2y}{2}\right]_0^{2-y} dy$$

$$= \frac{1}{2}\int_0^1 y^2\,dy + \frac{1}{2}\int_1^2 y(2-y)^2\,dy \qquad 4y+y^3-4y^2$$

$$= \frac{1}{2}\left(\frac{y^3}{3}\right)_0^1 + \frac{1}{2}\left[\frac{y^4}{4}-\frac{4y^3}{3}+2y^2\right]_1^2$$

$$= \frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{2}\left[4-\frac{32}{3}+8-\left(\frac{1}{4}-\frac{4}{3}+2\right)\right]$$

$$= \frac{1}{6}+\frac{1}{2}\left[\frac{4}{3}-\frac{9}{4}+\frac{4}{3}\right]$$

$$= \frac{1}{6}+\frac{1}{2}\left(\frac{5}{12}\right)$$

$$= \frac{1}{6}+\frac{5}{24} = \frac{3}{8} \text{ Ans.}$$

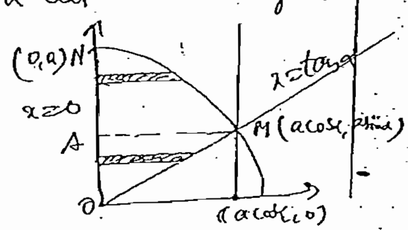
→ Change the order of integration in

$$\int_0^{2a}\int_{x^2/4a}^{3a-x} f(x,y)\,dx\,dy.$$

→ Change the order of integration in ~~the integral~~

$$\int_0^{a\cos\alpha}\int_{x\tan\alpha}^{\sqrt{a^2-x^2}} f(x,y)\,dx\,dy$$ and verify the result when $f(x,y)=1$.

$\boxed{\text{Ans: } \frac{a^2}{2}\left(\frac{\pi}{2}-\alpha\right)}$



7. Change the order of integration in $\int_0 \int_{x^2/a} V\,dx\,dy$ where V is a function of x and y.

Sol^n: The limits of integration are given by the parabolas $\frac{x^2}{a}=y$ i.e, $x^2=ay$;

$x-x^2/a=y$ i.e, $ax-x^2=ay$

and the lines $x=0$, $x=\frac{1}{2}a$.

Also the equation of parabola $ax-x^2=ay$ may be written as $\left(x-\frac{a}{2}\right)^2=-a\left(y-\frac{a}{4}\right)$

i.e, this parabola has the vertex as the point $\left(\frac{a}{2},\frac{a}{4}\right)$ and its concavity is downwards.

The points of intersection of two parabolas are given as follows:

$ax-x^2=x^2 \Rightarrow x(a-2x)=0$

$\Rightarrow x=0$ or $x=\frac{a}{2}$.

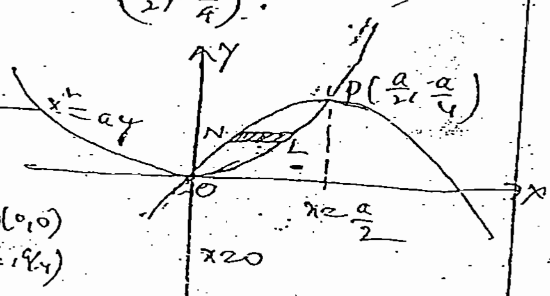
and hence from $x^2=ay$,

we get $y=0$ at $x=0$

$y=\frac{a}{4}$ at $x=\frac{a}{2}$.

Hence the points of intersection of the two parabola are $(0,0)$, and $\left(\frac{a}{2},\frac{a}{4}\right)$.

Draw the two parabolas $x^2=ay$ and $ax=x^2-ay$ intersecting at the points $O(0,0)$ and $P(a/2, a/4)$



Now draw the lines $x=0$ and ...

Clearly the integral extends to the area ONPLO.

Now take strips of the type NL parallel to the x-axis.

Solving $ay = ax - x^2$ for x, we get

$\therefore x^2 - ax + ay = 0$

$$x = \frac{1}{2}\left[a \pm \sqrt{a^2 - 4ay}\right]$$

$$= \frac{1}{2}\left[a - \sqrt{a^2 - 4ay}\right]$$

rejecting the +ve sign before square root since x is not greater than $\frac{a}{2}$ for the region of intersection.

For the region ONPLO, the strip NL has the extremities N and L on $ax - x^2 = ay$ and $x^2 = ay$.

Thus the limits of x are from $x = \frac{1}{2}[a - \sqrt{a^2 - 4ay}]$ to $x = \sqrt{ay}$.

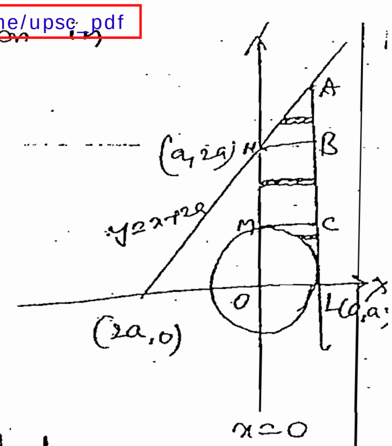
For limits of y, at O, $y = 0$ and at P $y = \frac{a}{4}$.

Hence changing the order of integration we have

$$\int_0^{a/2} \int_{x^2/a}^{x - x^2/a} V\,dx\,dy = \int_0^{a/4} \int_{\frac{1}{2}(a - \sqrt{a^2 - 4ay})}^{\sqrt{ay}} V\,dy\,dx$$

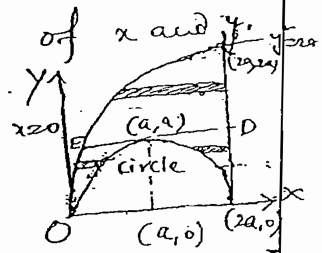
→ Change the order of integration in

$$\int_0^a \int_{\sqrt{a^2-x^2}}^{x+2a} \phi(x,y)\,dx\,dy.$$



→ Change the order of integration in the double integral $\int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} V\,dx\,dy$.

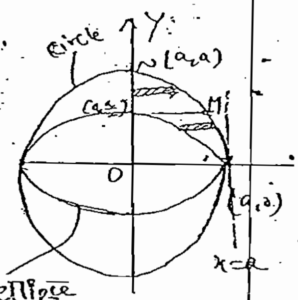
where V is a function of x and y.



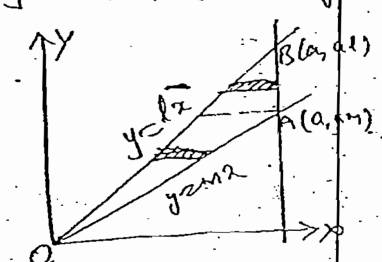
→ Change the order of integration in $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{ax}} \phi(x,y)\,dx\,dy$

→ change the order of integration in

$\int_0^a \int_{\frac{b}{a}\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} U\,dx\,dy$, where U is a function of x and y.



→ Change the order of integration in $\int_0^a \int_{mx}^{lx} V\,dx\,dy$ where V is a function of x and y.

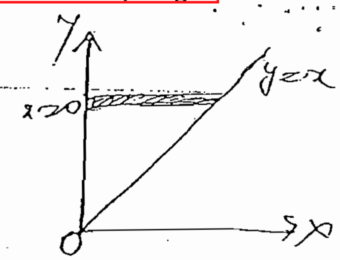


→ Show that

$$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} f(x,y)\,dx\,dy = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} f(x,y)\,dy\,dx$$

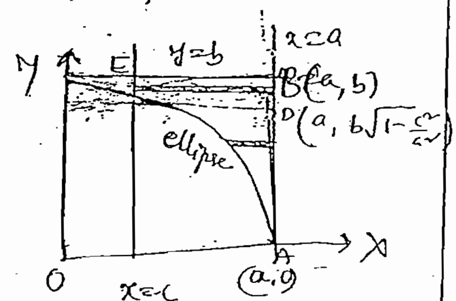
→ Change the order of integration in the double integral $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dx\,dy$ and hence find the value.

$\boxed{\text{Ans: } 1}$



→ Change the order of integration in $\int_c^a \int_{b/a\sqrt{a^2-x^2}}^{b} xy\,dx\,dy$

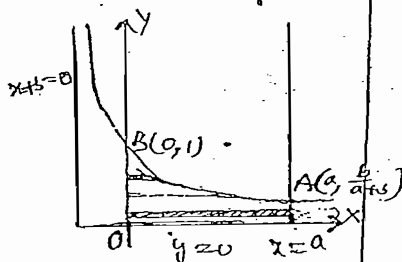
where c is less than a.



→ Change the order of integration in $\int_0^1 \int_0^{x(2-x)} f(x,y)\,dx\,dy$.

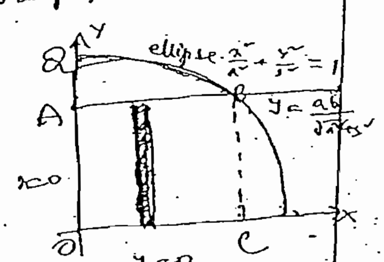
→ Change the order of integration in double integral

$$I = \int_0^a \int_0^{b/b+x} v\,dx\,dy.$$

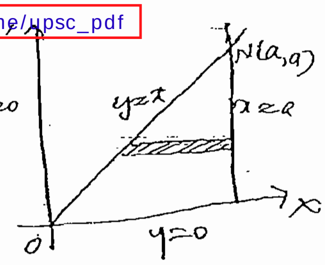


Ans: $I = \int_0^{b/a+b} \int_0^a v\,dy\,dx + \int_{b/a+b}^{1} \int_0^{\frac{b(1-y)}{y}} v\,dy\,dx.$

→ Change the order of integration in double integral $\int_0^{ab/\sqrt{a^2+b^2}} \int_a^{\frac{a}{b}\sqrt{b^2-y^2}}$



→ Change the order of integration in double integral $\int_0^a \int_0^x \frac{\phi'(y)\,dx\,dy}{\sqrt{(a-x)(x-y)}}$ and hence ~~find its~~ value.



The limits of integration are given by the straight lines $y=0$, $y=x$, $x=0$ and $x=a$.

Clearly the region of integration is ONM.

Take strips parallel to the x-axis.

The limits of x are from $x=y$ to $x=a$.

and the limits of y are from $y=0$ to $y=a$

Hence we have

$$\int_0^a\int_0^x \frac{\phi'(y)\,dx\,dy}{\sqrt{(a-x)(x-y)}} = \int_0^a\int_y^a \frac{\phi'(y)\,dy\,dx}{\sqrt{(a-x)(x-y)}}$$

To find the value:

Let $x = a\sin^2\theta + y\cos^2\theta$

$\Rightarrow dx = 2(a-y)\sin\theta\cos\theta\,d\theta.$

Also $a-x = a - a\sin^2\theta - y\cos^2\theta$

$= a\cos^2\theta - y\cos^2\theta = (a-y)\cos^2\theta.$

and $x-y = a\sin^2\theta + y\cos^2\theta - y$

$= a\sin^2\theta - y\sin^2\theta = (a-y)\sin^2\theta$

For limits of θ, when $x=y$, we have

$y = a\sin^2\theta + y\cos^2\theta$

$\Rightarrow (y-a)\sin^2\theta = 0$

i.e. $\sin\theta = 0$

i.e. $\theta = 0$

and when $x = a$, we have $a = 2a\sin^2\theta + y\cos^2\theta$

$\Rightarrow (x-y)\cos^2\theta = 0$

$\Rightarrow \cos\theta = 0$

$\Rightarrow \theta = \pi/2$

Thus the limits of θ are from $\theta = 0$ to $\theta = \pi/2$

$\therefore$ The given integral $= \int_0^a \int_y^a \frac{\phi'(y)\, dy\, dx}{\sqrt{(a-x)(x-y)}}$

$= \int_0^a \int_0^{\pi/2} \frac{\phi'(y)\, 2(a-y)}{(a-y)\sin\theta\cos\theta} \sin\theta\cos\theta\, dy\, d\theta$

$= 2\int_0^a \int_0^{\pi/2} \phi'(y)\, dy\, d\theta$

$= 2\int_0^a \phi'(y)\, [\theta]_0^{\pi/2}\, dy$

$= 2\cdot\frac{\pi}{2} \int_0^a \phi'(y)\, dy$

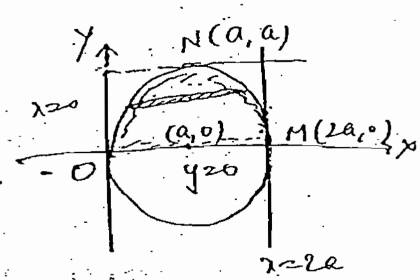
$= \pi\, [\phi(y)]_0^a$

$= \pi\,(\phi(a) - \phi(0))$

→ Change the order of integration in

$\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} \frac{\phi'(y)(x^2+y^2)x\, dx\, dy}{\sqrt{4a^2x^2-(x^2+y^2)^2}}$ and hence evaluate it.

Sol^n:



Evaluate $\int_0^{\pi/2}\int_0^{\pi/2} \sin x \sin^{-1}(\sin x \sin y)\, dx\, dy$.

Soln: Let $\sin x \sin y = \sin\theta$

Then $\sin x \cos y\, dy = \cos\theta\, d\theta$, keeping x constant.

When $y = 0$, $\sin\theta = 0 \Rightarrow \theta = 0$

& when $y = \pi/2$, $\sin\theta = \sin x \Rightarrow \theta = x$.

Hence θ varies from 0 to x.

$\therefore$ Given integral

$$\int_0^{\pi/2}\int_0^{\pi/2} \sin x \sin^{-1}(\sin x \sin y)\, dx\, dy$$

$$= \int_0^{\pi/2}\int_0^{x} \sin x \sin^{-1}(\sin\theta)\, dx \cdot \frac{\cos\theta\, d\theta}{\sin x \cos y}$$

$$= \int_0^{\pi/2}\int_0^{x} \sin x \cdot \theta \cdot dx\, \frac{\cos\theta}{\sin x \cos y}\, d\theta$$

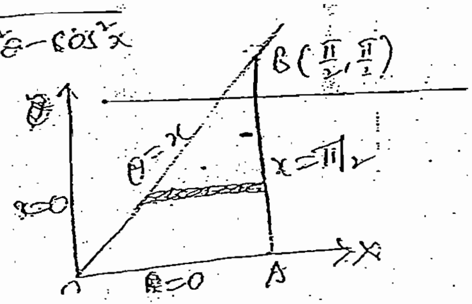
$$= \int_0^{\pi/2}\int_0^{x} \frac{\theta \cos\theta}{\cos y}\, dx\, d\theta$$

$$= \int_0^{\pi/2}\int_0^{x} \frac{\theta\cos\theta\, dx\, d\theta}{\sqrt{1-\frac{\sin^2\theta}{\sin^2 x}}} \quad \left(\because \sin x \sin y = \sin\theta \Rightarrow \sin y = \frac{\sin\theta}{\sin x} \Rightarrow \cos y = \sqrt{1-\sin^2 y}\right)$$

$$= \int_0^{\pi/2}\int_0^{x} \frac{\theta\cos\theta \sin x\, dx\, d\theta}{\sqrt{\sin^2 x - \sin^2\theta}}$$

$$= \int_0^{\pi/2}\int_0^{x} \frac{\theta\cos\theta \sin x\, dx\, d\theta}{\sqrt{\cos^2\theta - \cos^2 x}}$$

Clearly it is convenient to integrate first w.r.t x. Therefore we shall change



the order of integration.

The limits of integration are given by the straight lines $\theta = 0$, $\theta = x$ and $x = 0$, $x = \pi/2$.

Clearly the area of integration is OABO.

Consider strips parallel to x-axis.

The limits of x are from θ to $\pi/2$

and limits of θ are from 0 to $\pi/2$.

Hence changing the order of integral, we have

$$\int_0^{\pi/2}\int_0^{x} \frac{\theta\cos\theta\sin x\, dx\, d\theta}{\sqrt{\cos^2\theta - \cos^2 x}} = \int_0^{\pi/2}\int_{\theta}^{\pi/2} \frac{\theta\cos\theta \sin x}{\sqrt{\cos^2\theta - \cos^2 x}}\, d\theta\, dx$$

$$= \int_0^{\pi/2} \theta\cos\theta \left[-\sin^{-1}\left(\frac{\cos x}{\cos\theta}\right)\right]_{\theta}^{\pi/2} d\theta$$

$$= \int_0^{\pi/2} \theta\cos\theta \left[-0 + \frac{\pi}{2}\right] d\theta$$

$$= \frac{\pi}{2}\int_0^{\pi/2} \theta\cos\theta\, d\theta$$

$$= \frac{\pi}{2}\left[(\theta\sin\theta)_0^{\pi/2} - \int_0^{\pi/2}\sin\theta\, d\theta\right]$$

$$= \frac{\pi}{2}\left[\frac{\pi}{2} - 0 + [\cos\theta]_0^{\pi/2}\right]$$

$$= \frac{\pi}{2}\left[\frac{\pi}{2} - 1\right]$$

$$I = \int_{\theta}^{\pi/2} \frac{\sin x}{\sqrt{\cos^2\theta - \cos^2 x}}\, dx$$

let $\cos x = t \Rightarrow -\sin x\, dx = dt$; $t = \cos\theta$

limits $x = \theta \Rightarrow$; $x = \pi/2 \Rightarrow t = 0$.

$$\Rightarrow I = \int_{\cos\theta}^{0} \frac{-dt}{\sqrt{\cos^2\theta - t^2}}$$

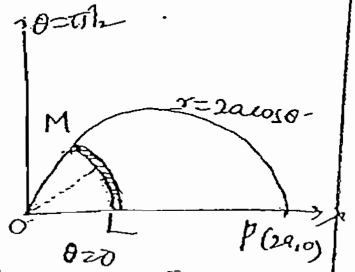
$$= -\sin^{-1}\left[\frac{t}{\cos\theta}\right]$$

$$= \left[-\sin^{-1}\left[\frac{\cos x}{\cos\theta}\right]\right]_{\theta}^{\pi/2}$$

Change of order of integration of polar co-ordinates :

→ Change the order of integration in double integral

$$\int_0^{\pi/2}\int_0^{2a\cos\theta} f(r,\theta)\, d\theta\, dr.$$



Sol: The limits of integration are given by $r=0$ (pole), $r=2a\cos\theta$ (a circle), $\theta=0$ (initial line) and $\theta=\pi/2$ (⊥lar to initial line at the pole)

Clearly the region of integration is OPMO.

To change the order of integration, we consider circular arc LM on which θ varies and r remains constant.

Now for limits of θ, the arc LM has its extremities on $\theta=0$ (initial line) and $r=2a\cos\theta$ (circle) i.e. $\theta=\cos^{-1}(r/2a)$

Also the limits of r are from $r=0$ to $r=2a$

Hence

$$\int_0^{\pi/2}\int_0^{2a\cos\theta} f(r,\theta)\, d\theta\, dr = \int_0^{2a}\int_0^{\cos^{-1}(r/2a)} f(r,\theta)\, dr\, d\theta$$

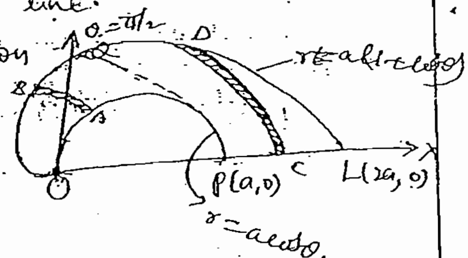
→ Change the order of integration in the system of integrals $\int_0^{\pi/2}\int_{a\cos\theta}^{a(1+\cos\theta)} f(r,\theta)\, r\, d\theta\, dr + \int_{\pi/2}^{\pi}\int_{a\cos\theta}^{a(1+\cos\theta)} f(r,\theta)\, r\, d\theta\, dr$

Sol: The limits of integration are given by from $r=a\cos\theta$ to $r=a(1+\cos\theta)$, $\theta=0$ to $\theta=\pi/2$ and $\theta=\pi/2$ to $\theta=\pi$.

i.e, the region of integration is bounded by

upper half circle $r = a\cos\theta$, upper half Cardioid $r = a(1+\cos\theta)$ and the initial line.

Clearly OAPLQO is the region of integration.



Now to change the order of integration consider elementary circular arcs (lines AB and RD) about pole 'O' as a centre.

These arcs change their character at P.

Hence the region is divided into two parts namely OAPQBO and QPLQ.

In the region OAPQBO, the extremities of the arc AB lie on $r = a\cos\theta$ and $r = a(1+\cos\theta)$

Hence θ varies from $\theta = \cos^{-1}(r/a)$ to $r = \cos^{-1}\left(\frac{r-a}{a}\right)$

Also r varies from $r=0$ to $r=a$ as $OP=a$.

In the region QPLQ, the extremities of the arc CD lie on $\theta=0$ and cardioid $r = a(1+\cos\theta)$.

Hence θ varies from $\theta=0$ to $\theta = \cos^{-1}\frac{r-a}{a}$,

and r varies from $r=a$ to $r=2a$ as $OL=2a$

Hence the given integral becomes

$$\int_0^a \int_{\cos^{-1}(r/a)}^{\cos^{-1}\left(\frac{r-a}{a}\right)} f(r,\theta)\, r\, dr\, d\theta + \int_a^{2a} \int_0^{\cos^{-1}\left(\frac{r-a}{a}\right)} f(r,\theta)\, r\, dr\, d\theta$$

Set - X

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Multiple Integrals and Their Applications

[illegible]

7.1. DOUBLE INTEGRALS

The definite integral $\int_a^b f(x)\,dx$ is defined as the limit of the sum

$$f(x_1)\delta x_1 + f(x_2)\delta x_2 + \ldots\ldots + f(x_n)\delta x_n,$$

where $n \to \infty$ and each of the lengths $\delta x_1, \delta x_2, \ldots\ldots$ tends to zero. A double integral is its counterpart in two dimensions.

Consider a function $f(x, y)$ of the independent variables x, y defined at each point in the finite region R of the xy-plane. Divide R into n elementary areas $\delta A_1, \delta A_2, \ldots\ldots, \delta A_n$. Let (x_r, y_r) be any point within the rth elementary area δA_r. Consider the sum

$$f(x_1, y_1)\,\delta A_1 + f(x_2, y_2)\,\delta A_2 + \ldots\ldots + f(x_n, y_n)\,\delta A_n, \text{ i.e. } \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r$$

The limit of this sum, if it exists, as the number of sub-divisions increases indefinitely and area of each sub-division decreases to zero, is defined as the *double integral of $f(x, y)$ over the region R* and is written as $\iint_R f(x, y)\,dA$.

Thus $$\iint_R f(x, y)\,dA = \underset{\substack{n \to \infty \\ \delta A \to 0}}{\text{Lt}} \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r \qquad \ldots(1)$$

The utility of double integrals would be limited if it were required to take limit of sums to evaluate them. However, there is another method of evaluating double integrals by successive single integrations.

For purposes of evaluation, (1) is expressed as the repeated integral $\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y)\,dxdy$ [illegible]

Its value is found as follows :

(i) *When y_1, y_2 are functions of x and x_1, x_2 are constants,* $f(x, y)$ is first integrated w[illegible] keeping x fixed between limits y_1, y_2 and then the resulting expression is integrated w[illegible] within the limits x_1, x_2 i.e.

$$I_1 = \int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} f(x, y)dy} \, dx$$

where integration is carried from the inner to the outer rectangle.

Fig. 7·1 illustrates this process. Here AB and CD are the two curves whose equations are $y_1 = f_1(x)$ and $y_2 = f_2(x)$. PQ is a vertical strip of width dx.

Then the inner rectangle integral means that the integration is along one edge of the strip PQ from P to Q (x remaining constant), while the outer rectangle integral corresponds to the sliding of the edge from AC to BD.

Thus the whole region of integration is the area $ABDC$.

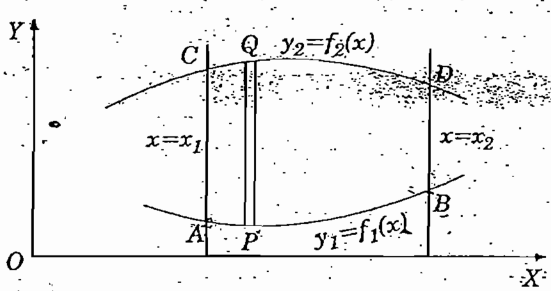

Fig. 7·1.

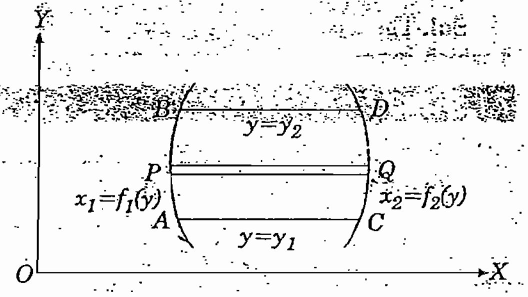

Fig. 7·2.

(ii) *When x_1, x_2 are functions of y and y_1, y_2 are constants*, $f(x, y)$ is first integrated w.r.t. x keeping y fixed, within the limits x_1, x_2 and the resulting expression is integrated w.r.t. y between the limits y_1, y_2, *i.e.*

$$I_2 = \int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x, y)dx} \, dy$$ which is geometrically illustrated by Fig. 7·2.

Here AB and CD are the curves $x_1 = f_1(y)$ and $x_2 = f_2(y)$. PQ is a horizontal strip of width dy.

Then inner rectangle indicates that the integration is along one edge of this strip from P to Q while the outer rectangle corresponds to the sliding of this edge from AC to BD.

Thus the whole region of integration is the area $ABDC$.

(iii) *When both pairs of limits are constants*, the region of integration is the rectangle $ABDC$ (Fig. 7·3).

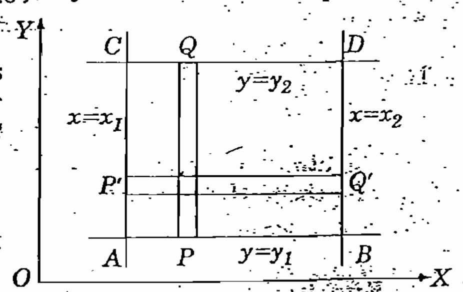

Fig. 7·3.

In I_1, we integrate along the vertical strip PQ and then slide it from AC to BD.

In I_2, we integrate along the horizontal strip $P'Q'$ and then slide it from AB to CD.

Here obviously $I_1 = I_2$.

Thus *for constant limits, it hardly matters whether we first integrate w.r.t. x and then w.r.t. y or vice versa.*

Example 7·1. *Evaluate* $\int_0^5 \int_0^{x^2} x(x^2+y^2)dxdy$.

Sol. $I = \int_0^5 dx \int_0^{x^2} (x^3+xy^2)dy = \int_0^5 \left[x^3y + x \cdot \frac{y^3}{3}\right]_0^{x^2} dx = \int_0^5 \left[x^3 \cdot x^2 + x \cdot \frac{x^6}{3}\right] dx$

$$= \int_0^5 \left(x^5 + \frac{x^7}{3}\right)dx = \left|\frac{x^6}{6} + \frac{x^8}{24}\right|_0^5 = 5^6\left[\frac{1}{6} + \frac{5^2}{24}\right] = 18880\cdot2 \text{ nearly.}$$

Example 7·2. *Evaluate* $\iint_A xy\, dx\, dy$, *where A is the domain bounded by x-axis, ordinate* $x = 2a$ *and the curve* $x^2 = 4ay$. *(Gulbarga, 1999 S)*

Sol. The line $x = 2a$ and the parabola $x^2 = 4ay$ intersect at $L(2a, a)$. Fig. 7·4 shows the domain A which is the area OML.

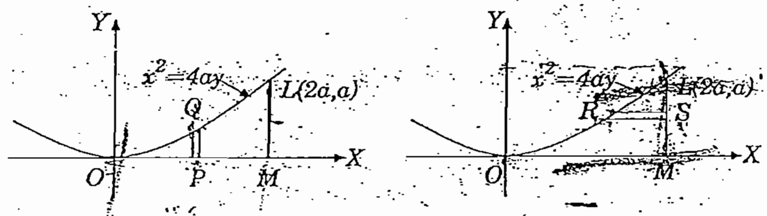

Fig. 7·4.

Integrating first over a vertical strip PQ, *i.e.* w.r.t. y from $P(y = 0)$ to $Q(y = x^2/4a)$ on the parabola and then w.r.t. x from $x = 0$ to $x = 2a$, we have

$$\iint_A xy\, dx\, dy = \int_0^{2a} dx \int_0^{x^2/4a} xy\, dy = \int_0^{2a} x\left[\frac{y^2}{2}\right]_0^{x^2/4a} dx$$

$$= \frac{1}{32a^2}\int_0^{2a} x^5\, dx = \frac{1}{32a^2}\left|\frac{x^6}{6}\right|_0^{2a} = \frac{a^4}{3}.$$

Otherwise integrating first over a horizontal strip RS, *i.e.* w.r.t. x from $R(x = 2\sqrt{ay})$ on the parabola to $S(x = 2a)$ and then w.r.t. y from $y = 0$ to $y = a$, we get

$$\iint_A xy\, dx\, dy = \int_0^a dx \int_{2\sqrt{(ay)}}^{2a} xy\, dx = \int_0^a y\left[\frac{x^2}{2}\right]_{2\sqrt{(ay)}}^{2a} dy$$

$$= 2a\int_0^a (ay - y^2)dy = 2a\left[\frac{ay^2}{2} - \frac{y^3}{3}\right]_0^a = \frac{a^4}{3}.$$

7·2. CHANGE OF ORDER OF INTEGRATION

In a double integral with variable limits, the change of order of integration changes the limits of integration. While doing so, sometimes it is required to split up the region of integration and the given integral is expressed as the sum of a number of double integrals with changed limits. To fix up the new limits, it is always advisable to draw a rough sketch of the region of integration.

The change of order of integration quite often facilitates the evaluation of a double integral. The following examples will make these ideas clear.

Example 7·3. *Change the order of integration in the integral*

$$I = \int_{-a}^{a} \int_{0}^{\sqrt{(a^2 - y^2)}} f(x, y)\, dx\, dy.$$

Sol. Here the elementary strip is parallel to x-axis (such as PQ) and extends from $x = 0$ to $x = \sqrt{(a^2 - y^2)}$ (*i.e.* to the circle $x^2 + y^2 = a^2$) and this strip slides from $y = -a$ to $y = a$. This shaded semi-circular area is, therefore, the region of integration (Fig. 7·5).

On changing the order of integration, we first integrate w.r.t. y along a vertical strip RS which extends from R $[y = -\sqrt{(a^2 - x^2)}]$ to S $[y = \sqrt{(a^2 - x^2)}]$. To cover the given region, we then integrate w.r.t. x from $x = 0$ to $x = a$.

Thus $\quad I = \int_{0}^{a} dx \int_{-\sqrt{(a^2 - x^2)}}^{\sqrt{(a^2 - x^2)}} f(x, y)\, dy$

or $\quad = \int_{0}^{a} \int_{-\sqrt{(a^2 - x^2)}}^{\sqrt{(a^2 - x^2)}} f(x, y)\, dy\, dx.$

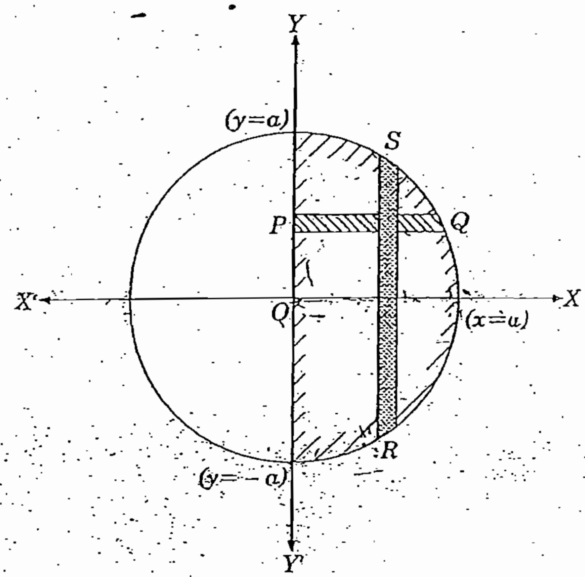

Fig. 7·5.

Example 7·4. *Change the order of integration in* $I = \int_{0}^{1} \int_{x^2}^{2-x} xy\, dx\, dy$ *and hence evaluate the same.* (Andhra, 1999 ; Gauhati, 1999)

Sol. Here the integration is first w.r.t. y along a vertical strip PQ which extends from P on the parabola $y = x^2$ to Q on the line $y = 2 - x$. Such a strip slides from $x = 0$ to $x = 1$, giving the region of integration as the curvilinear triangle OAB (shaded) in Fig. 7·6.

On changing the order of integration, we first integrate w.r.t. x along a horizontal strip $P'Q'$ and that requires the splitting up of the region OAB into two parts by the line $AC(y = 1)$, *i.e.* the curvilinear triangle OAC and the triangle ABC.

For the region OAC, the limits of integration for x are from $x = 0$ to $x = \sqrt{y}$ and those for y are from $y = 0$ to $y = 1$. So the contribution to I from the region OAC is

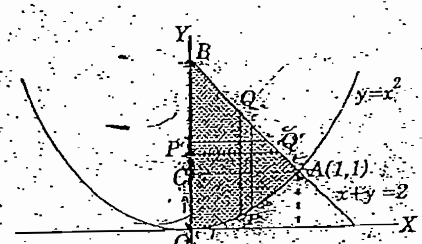

Fig. 7·6.

$$I_1 = \int_{0}^{1} dy \int_{0}^{\sqrt{y}} xy\, dx.$$

For the region ABC, the limits of integration for x are from $x = 0$ to $x = 2 - y$ and those for y are from $y = 1$ to $y = 2$. So the contribution to I from the region ABC is

$$I_2 = \int_{1}^{2} dy \int_{0}^{2-y} xy\, dx.$$

Hence, on reversing the order of integration,

$$I = \int_0^1 dy \int_0^{\sqrt{y}} xy\,dx + \int_1^2 dy \int_0^{2-y} xy\,dx$$

$$= \int_0^1 dy \left| \frac{x^2}{2} \cdot y \right|_0^{\sqrt{y}} + \int_1^2 dy \left| \frac{x^2}{2} \cdot y \right|_0^{2-y} = \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 y(2-y)^2 dy$$

$$= \frac{1}{6} + \frac{5}{24} = \frac{3}{8}.$$

7.3. DOUBLE INTEGRALS IN POLAR CO-ORDINATES

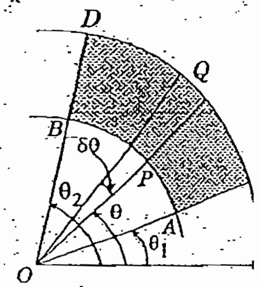

Fig. 7.7.

To evaluate $\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r, \theta)\, dr\, d\theta$, we first integrate w.r.t. r between limits $r = r_1$ and $r = r_2$ keeping θ fixed and the resulting expression is integrated w.r.t. θ from θ_1 to θ_2. In this integral, r_1, r_2 are functions of θ and θ_1, θ_2 are constants.

Fig. 7.7 illustrates the process geometrically.

Here AB and CD are the curves $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$ bounded by the lines $\theta = \theta_1$ and $\theta = \theta_2$. PQ is a wedge of angular thickness $\delta\theta$.

Then $\int_{r_1}^{r_2} f(r, \theta)\, dr$ indicates that the integration is along PQ from P to Q while th integration w.r.t. θ corresponds to the turning of PQ from AC to BD.

Thus the whole region of integration is the area $ACDB$. The order of integration may b changed with appropriate changes in the limits.

Example 7.5. *Calculate* $\iint r^3\, dr\, d\theta$ *over the area included between the circles* $r = 2 \sin \theta$ *an* $r = 4 \sin \theta$. (*J.N.T.U., 1999; Marathwada, 199*

Sol. Given circles $r = 2 \sin \theta$ ...(i

and $r = 4 \sin \theta$ ...(ii

are shown in Fig. 7.8. The shaded area between these circles is the region of integration.

If we integrate first w.r.t. r, then its limits are from $P(r = 2 \sin \theta)$ to $Q(r = 4 \sin \theta)$ and to cover the whole region θ varies from 0 to π. Thus the required integral is

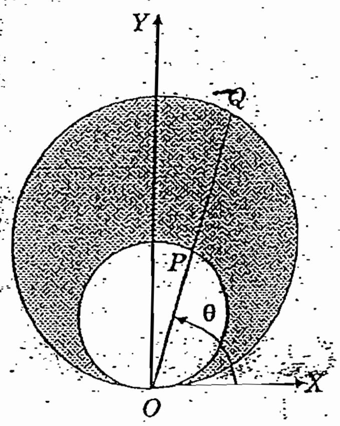

Fig. 7.8.

$$I = \int_0^{\pi} d\theta \int_{2\sin\theta}^{4\sin\theta} r^3 dr = \int_0^{\pi} d\theta \left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta}$$

$$= 60 \int_0^{\pi} \sin^4\theta\, d\theta = 60 \times 2 \int_0^{\pi/2} \sin^4\theta\, d\theta$$

$$= 120 \times \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} = 22.5\,\pi.$$

Problems 7.1

Evaluate the following integrals (1—7) :

1. $\int_1^2 \int_1^3 xy^2\,dx\,dy$ *(Madras, 1998 S)*

2. $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)dx\,dy$. *(V.T.U., 2000)*

3. $\int_0^4 \int_0^{x^2} e^{y/x}\,dy\,dx$. *(Osmania, 1999 S)*

4. $\int_0^1 \int_0^{\sqrt{(1+x^2)}} \frac{dy\,dx}{1+x^2+y^2}$ *(Madras, 2000)*

5. $\int\int xy\,dx\,dy$ over the positive quadrant of the circle $x^2+y^2=a^2$. *(V.T.U., 2001 ; Madras, 2000)*

6. $\iint (x+y)^2\,dxdy$ over the area bounded by the ellipse $x^2/a^2+y^2/b^2=1$.

7. $\int\int xy(x+y)\,dxdy$ over the area between $y=x^2$ and $y=x$.

Evaluate the following integrals by changing the order of integration (8—16) :

8. $\int_0^1 \int_0^{\sqrt{(1-x^2)}} y^2dxdy$. *(Pondicherry, 1998S)*

9. $\int_0^3 \int_1^{\sqrt{(4-y)}} (x+y)\,dxdy$. *(Anna, 2003 S ; V.T.U., 2003 ; Delhi, 2002)*

10. $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$

11. $\int_0^1 \int_x^{\sqrt{(2-x^2)}} \frac{x\,dy\,dx}{\sqrt{(x^2+y^2)}}$ *(Rohtak, 2003 ; I.S.M., 2001)*

12. $\int_0^{a/\sqrt{2}} \int_y^{\sqrt{(a^2-y^2)}} \log(x^2+y^2)\,dx\,dy\ (a>0)$. *(Bhopal, 1998)*

13. $\int_0^a \int_{x/a}^{\sqrt{(x/a)}} (x^2+y^2)\,dx\,dy$. *(Marathwada, 1998)*

14. $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2\,dy\,dx}{\sqrt{y^4-a^2x^2}}$

15. $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$. *(Madras, 2003 ; V.T.U., 2000)*

16. $\int_0^\infty \int_0^x xe^{-x^2/y}\,dy\,dx$. *(V.T.U., 2004 ; Delhi, 2002)*

17. Sketch the region of integration of $\int_a^{ae^{\pi/4}} \int_{2\log(r/a)}^{\pi/2} f(r,\theta)\,r\,dr\,d\theta$ and change the order of integration.

18. Evaluate $\iint r\sin\theta\,dr\,d\theta$ over the cardioid $r=a(1-\cos\theta)$ above the initial line.

19. Show that $\iint_R r^2\sin\theta\,dr\,d\theta = 2a^2/3$, where R is the semi-circle $r=2a\cos\theta$ above the initial line.

20. Evaluate $\int\int \frac{rdrd\theta}{\sqrt{a^2+r^2}}$ over one loop of the lemniscate $r^2=a^2\cos 2\theta$.

7.4. AREA ENCLOSED BY PLANE CURVES

(1) *Cartesian co-ordinates.*

Consider the area enclosed by the curves $y=f_1(x)$ and $y=f_2(x)$ and the ordinates $x=x_1, x=x_2$ (Fig. 7·9).

Divide this area into vertical strips of width δx. If $P(x, y)$, $Q(x+\delta x, y+\delta y)$ be two neighbouring points, then the area of the small rectangle $PQ = \delta x \delta y$.

$\therefore$ area of strip $KL = \underset{\delta y \to 0}{\text{Lt}} \ \Sigma \delta x \ \delta y$.

Since for all rectangles in this strip δx is the same and y varies from $y = f_1(x)$ to $y = f_2(x)$.

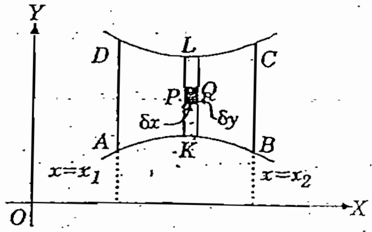

Fig. 7·9.

$$\therefore \text{ area of the strip } KL = \delta x \underset{\delta y \to 0}{\text{Lt}} \sum_{f_1(x)}^{f_2(x)} dy = \delta x \int_{f_1(x)}^{f_2(x)} dy.$$

Now adding up all such strips from $x = x_1$ to $x = x_2$, we get the area $ABCD$

$$= \underset{\delta x \to 0}{\text{Lt}} \sum_{x_1}^{x_2} \delta x \cdot \int_{f_1(x)}^{f_2(x)} dy = \int_{x_1}^{x_2} dx \int_{f_1(x)}^{f_2(x)} dy$$

$$= \int_{x_1}^{x_2} \int_{f_1(x)}^{f_2(x)} dx\, dy$$

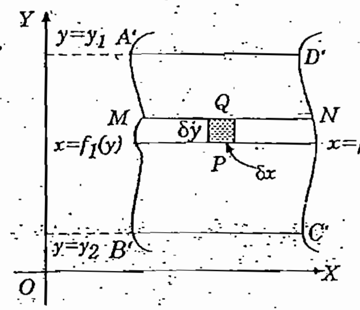

Fig. 7·10.

Similarly dividing the area $A'B'C'D'$ (Fig. 7·10) into horizontal strips of width δy, we get the area $A'B'C'D'$.

$$= \int_{y_1}^{y_2} \int_{f_1(y)}^{f_2(y)} dx\, dy$$

(2) *Polar co-ordinates.*

Consider an area A enclosed by a curve whose equation is in polar co-ordinates.

Let $P(r, \theta)$, $Q(r + \delta r, \ \theta + \delta\theta)$ be two neighbouring points. Mark circular areas of radii r and $r + \delta r$ meeting OQ in R and OP (produced) in S (Fig. 7·11).

Since arc $PR = r\delta\theta$ and $PS = \delta r$.

$\therefore$ area of the curvilinear rectangle $PRQS$ is approximately $= PR \cdot PS = r\delta\theta \cdot \delta r$.

If the whole area is divided into such curvilinear rectangles, the sum $\Sigma\Sigma r\delta\theta\delta r$ taken for all these rectangles, gives in the limit the area A.

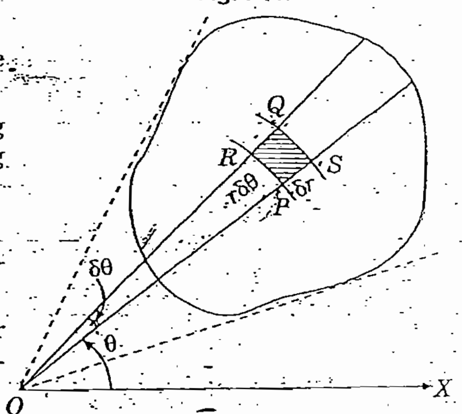

Fig. 7·11.

Hence $\quad A = \underset{\substack{\delta r \to 0 \\ \delta\theta \to 0}}{\text{Lt}} \Sigma\Sigma r\delta\theta\delta r = \int\int r d\theta dr$

where the limits are to be so chosen as to cover the entire area.

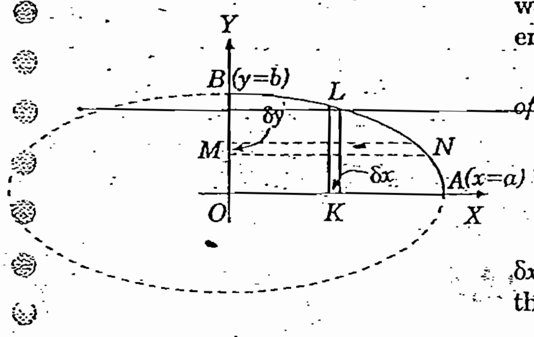

Fig. 7·12.

Example 7·6. *Find the area of a plate in the form of a quadrant of the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad \textit{(V.T.U., 2001 ; Osmania, 2000 S)}$$

Sol. Dividing the area into vertical strips of width δx, y varies from $K(y = 0)$ to $L\,[y = b\sqrt{(1 - x^2/a^2)}]$ and then x varies from 0 to a (Fig. 7·12).

Sol. Changing to polar spherical coordinates by putting

$$x = r\sin\theta\cos\phi,\; y = r\sin\theta\sin\phi,\; z = r\cos\theta$$

we have $dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$.

Also the volume of the sphere is 8 times the volume of its portion in the positive octant for which r varies from 0 to a, θ varies from 0 to $\pi/2$ and ϕ varies from 0 to $\pi/2$.

$\therefore$ Volume of the sphere

$$= 8\int_0^a \int_0^{\pi/2} \int_0^{\pi/2} r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 8\int_0^a r^2\,dr \cdot \int_0^{\pi/2} \sin\theta\,d\theta \cdot \int_0^{\pi/2} d\phi = 8\cdot\left|\frac{r^3}{3}\right|_0^a \cdot \left|-\cos\theta\right|_0^{\pi/2} \cdot \frac{\pi}{2}$$

$$= 4\pi\cdot\frac{a^3}{3}\cdot(-0+1) = \frac{4}{3}\pi a^3.$$

Example 7·18. *Find the volume of the portion of the sphere $x^2+y^2+z^2=a^2$ lying inside the cylinder $x^2+y^2=ay$.* *(Rohtak, 2003)*

Sol. The required volume is easily found by changing to cylindrical co-ordinates (ρ, ϕ, z). We therefore have

$$x = \rho\cos\phi,\; y = \rho\sin\phi,\; z = z$$

and
$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho.$$

Then the equation of the sphere becomes $\rho^2 + z^2 = a^2$ and that of cylinder becomes $\rho = a\sin\phi$.

The volume inside the cylinder bounded by the sphere is twice the volume shown shaded in the Fig. 7·23 for which z varies from 0 to $\sqrt{(a^2-\rho^2)}$, ρ varies from 0 to $a\sin\phi$ and ϕ varies from 0 to π.

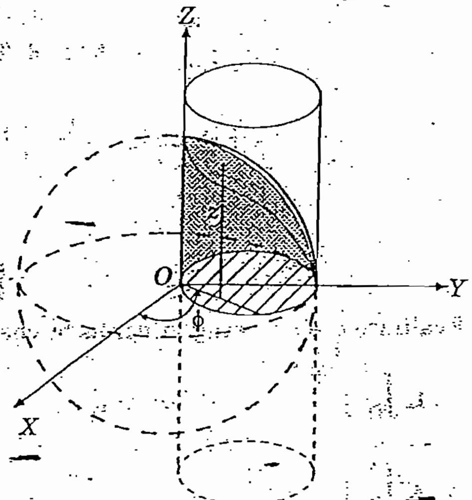

Fig. 7·23.

Hence the required volume

$$= 2\int_0^{\pi} \int_0^{a\sin\phi} \int_0^{\sqrt{(a^2-\rho^2)}} \rho\,dz\,d\rho\,d\phi$$

$$= 2\int_0^{\pi} \int_0^{a\sin\phi} \rho\sqrt{(a^2-\rho^2)}\,d\rho\,d\phi = 2\int_0^{\pi} \left|-\frac{1}{3}(a^2-\rho^2)^{3/2}\right|_0^{a\sin\phi} d\phi$$

$$= \frac{2a^3}{3}\int_0^{\pi} (1-\cos^3\phi)\,d\phi = \frac{2a^3}{9}(3\pi-4).$$

Example 7·19. *Evaluate* $\displaystyle\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_{\sqrt{(x^2+y^2)}}^1 \frac{dz\,dy\,dx}{\sqrt{(x^2+y^2+z^2)}}$

Sol. We change to spherical polar co-ordinates (r, θ, ϕ), so that

$$x = r\sin\theta\cos\phi,\; y = r\sin\theta\sin\phi,\; z = r\cos\theta$$

and
$$J = r^2\sin\theta,\; x^2+y^2+z^2 = r^2.$$

The region of integration is common to the cone $z^2 = x^2 + y^2$ and the cylinder $x^2 + y^2 = 1$ bounded by the plane $z = 1$ in the positive octant (Fig. 7·24). Hence θ varies from 0 to $\pi/4$, r varies from 0 to $\sec\theta$ and ϕ varies from 0 to $\pi/2$.

$\therefore$ Given integral becomes

$$\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sec\theta}\frac{1}{r}\cdot r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$=\int_0^{\pi/2} d\phi\int_0^{\pi/4}\left|\frac{r^2}{2}\right|_0^{\sec\theta}\sin\theta\,d\theta$$

$$=\frac{\pi}{2}\int_0^{\pi/4}\frac{\sec^2\theta}{2}\sin\theta\,d\theta$$

$$=\frac{\pi}{4}\int_0^{\pi/4}\sec\theta\tan\theta\,d\theta$$

$$=\frac{\pi}{4}\left|\sec\theta\right|_0^{\pi/4}=\frac{(\sqrt{2}-1)\pi}{4}$$

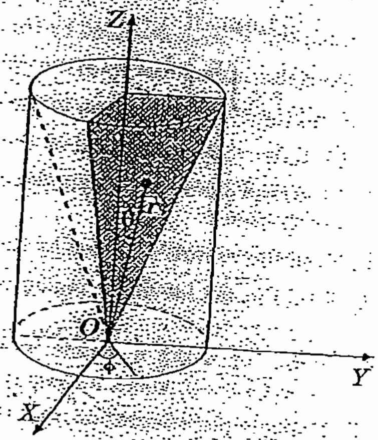

Fig. 7·24

Problems 7.4

Evaluate the following integrals by changing to polar co-ordinates :

1. $\int_0^a\int_0^{\sqrt{(a^2-y^2)}}(x^2+y^2)\,dy\,dx.$

2. $\int_0^a\int_y^a\frac{x^2\,dx\,dy}{\sqrt{(x^2+y^2)}}$. *(Andhra, 1998 ; Delhi, 1997)*

3. $\int_0^{4a}\int_{y^2/4a}^{y}\frac{x^2-y^2}{x^2+y^2}\,dx\,dy$ *(Marathwada, 1998)*

4. $\iint xy\,(x^2+y^2)^{n/2}\,dx\,dy$ over the positive quadrant of $x^2+y^2=4$, supposing $n+3>0$.

5. Transform the following to cartesian form and hence evaluate $\int_0^{\pi}\int_0^{a} r^3\sin\theta\cos\theta\,dr\,d\theta$.

6. By using the transformation $x+y=u$, $y=uv$, show that $\int_0^1\int_0^{1-x} e^{y/(x+y)}\,dy\,dx=\frac{1}{2}(e-1)$.

Evaluate the following integrals by changing to spherical co-ordinates :

7. $\int_0^1\int_0^{\sqrt{(1-x^2)}}\int_0^{\sqrt{(1-x^2-y^2)}}\frac{dx\,dy\,dz}{\sqrt{(1-x^2-y^2-z^2)}}$

(Madras, 1998 ; Marathwada, 1998 S ; Punjab, 1997)

8. $\iiint z^2\,dx\,dy\,dz$, taken over the volume bounded by the surfaces $x^2+y^2=a^2$, $x^2+y^2=z$ and $z=0$. (J.N.T.U., 199

9. Find the volume bounded by the xy-plane, the cylinder $x^2+y^2=1$ and the plane $x+y+z=3$. (I.S.M., 200

10. Find the volume bounded by the xy-plane, the paraboloid $2z=x^2+y^2$ and the cylinder $x^2+y^2=4$.

R-2004 11. Find the volume cut from the sphere $x^2+y^2+z^2=a^2$ by the cone $x^2+y^2=z^2$.

12. Find the volume common to the cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$. (Delhi, 199

13. Find the volume enclosed by the cylinders $x^2+y^2=2ax$ and $z^2=2ax$.

14. Find the volume of the cylinder $x^2+y^2-2ax=0$, intercepted between the paraboloid $x^2+y^2=2az$ ar the xy-plane.

15. Find the volume bounded by the cylinder $x^2+y^2=4$ and the hyperboloid $x^2+y^2-z^2=1$.

16. Find the volume of the region bounded by $z=x^2+y^2$, $z=0$, $x=-a$, $x=a$ and $y=-a$, $y=a$. (Kanpur, 199

17. Prove, by using a double integral that the volume generated by the revolution of the cardio $r=a(1+\cos\theta)$ about its axis is $8\pi a^3/3$. (V.T.U., 200

2006 P-I 18. Using triple integration, find the volume of the sphere $x^2+y^2+z^2=a^2$. (V.T.U., 200

19. Evaluate $\iiint (x+y+z)\,dx\,dy\,dz$ over the tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$ ar $x+y+z=1$. [See Fig. 7·28] (Bangalore, 1998

20. Find the volume of the tetrahedron bounded by the co-ordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. (Burdwan, 200

21. *Find the volume of the solid surrounded by the surface $(x/a)^{2/3}+(y/b)^{2/3}+(z/c)^{2/3}=1$.*

[Sol. Changing the variables, x, y, z to X, Y, Z where

$$(x/a)^{1/3}=X,\ (y/b)^{1/3}=Y,\ (z/c)^{1/3}=Z$$

i.e. $$x=aX^3,\ y=bY^3,\ z=cZ^3 \text{ so that}$$

$$J=\partial(x,y,z)/\partial(X,Y,Z)=27\,abc\,X^2Y^2Z^2.$$

$$\therefore \text{ Reqd. volume}=\iiint dx\,dy\,dz=27\,abc\iiint X^2Y^2Z^2\,dX\,dY\,dZ$$

taken throughout the sphere $X^2+Y^2+Z^2=1$.

Now change X, Y, Z to spherical polar coordinates r, θ, ϕ so that $X=r\sin\theta\cos\phi$, $Y=r\sin\theta\sin$ $Z=r\cos\theta$, and $\partial(X,Y,Z)/\partial(r,\theta,\phi)=r^2\sin\theta$. To describe the positive octant of the sphere (i), r varies fro 0 to 1, θ from 0 to $\pi/2$ and ϕ from 0 to $\pi/2$.

$$\therefore \text{ Reqd. volume}=27\,abc\times 8\int_0^1\int_0^{\pi/2}\int_0^{\pi/2} r^2\sin^2\theta\cos^2\phi\times r^2\sin^2\theta\sin^2\phi\cdot r^2\cos^2\theta\cdot r^2\sin\theta\,dr\,d\theta\ldots$$

$$=216\,abc\int_0^1 r^8\,dr\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta\int_0^{\pi/2}\sin^2\phi\cos^2\phi\,d\phi=4\pi abc/35]$$

22. Work out example 7·12 by changing the variables.

7·8. AREA OF A CURVED SURFACE

Consider a point P of the surface $S: z=f(x,y)$.

Let its projection on the xy-plane be the region A. Divide it into area elements by drawi lines parallel to the axes of X and Y. (Fig. 7·25).

On the element $\delta x \delta y$ as base, erect a cylinder having generators parallel to OZ and meeting the surface S in an element of area δS.

As $\delta x \delta y$ is the projection of δS on the xy plane,

$\therefore \delta x\, \delta y = \delta S \cdot \cos \gamma$, where γ is the angle between the xy-plane and the tangent plane to S at P, *i.e.* it is the angle between the Z-axis and the normal to S at $P (= \angle Z'PN)$.

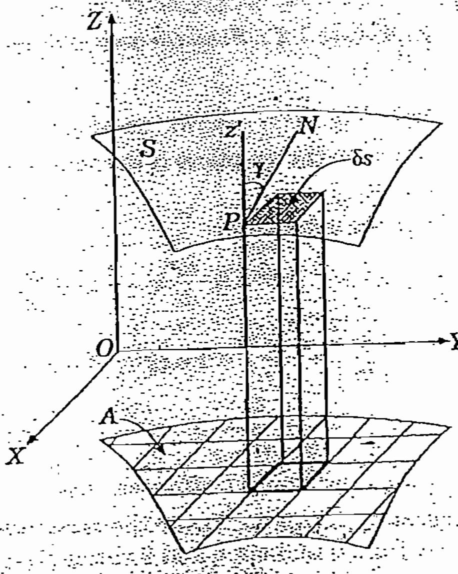

Fig. 7·25.

Now since the direction cosines of the normal to the surface $F(x, y, z) = 0$ are proportional to

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}.$$

$\therefore$ the direction cosines of the normal to $S [F = f(x, y) - z]$ are proportional to $-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1$ and those of the z-axis are 0, 0, 1.

Hence $$\cos \gamma = \frac{1}{\sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}} \quad \therefore \quad \delta S = \frac{\delta x\, \delta y}{\cos \gamma} = \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\, \delta x\, \delta y$$

Hence $$S = \operatorname*{Lt}_{\delta S \to 0} \sum \delta S = \iint_A \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\, dx\, dy$$

Similarly, if B and C be the projections of S on the yz-and zx-planes respectively, then

$$S = \iint_B \sqrt{\left[\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1\right]}\, dy\, dz \text{ and } S = \iint_C \sqrt{\left[\left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2 + 1\right]}\, dz\, dx.$$

Example 7·20. *Find the area of the portion of the cylinder $x^2 + z^2 = 4$ lying inside the cylinder $x^2 + y^2 = 4$.*

Sol. Fig. 7·26 shows one-eighth of the required area. Its projection on the xy-plane is a quadrant circle $x^2 + y^2 = 4$.

For the cylinder $x^2 + z^2 = 4$, ...(*i*)

we have $\frac{\partial z}{\partial x} = -\frac{x}{z}, \frac{\partial z}{\partial y} = 0$.

so that $$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 = \frac{x^2 + z^2}{z^2} = \frac{4}{4 - x^2}.$$

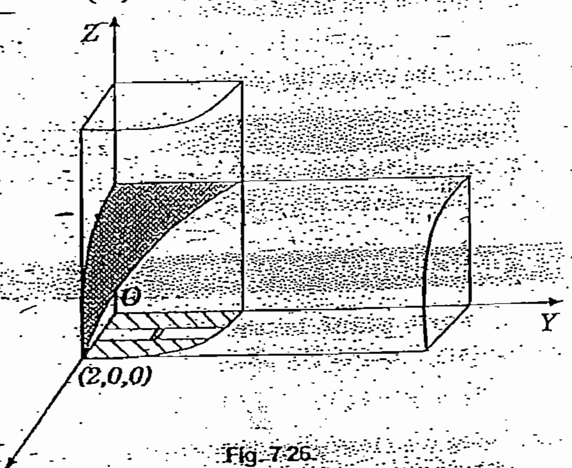

Fig. 7·26.

Hence the required surface area = 8 (surface area of the upper portion of (*i*) lying within the cylinder $x^2 + y^2 = 4$ in the positive octant)

$$= 8 \int_0^2 \int_0^{\sqrt{(4 - x^2)}} \frac{2}{\sqrt{(4 - x^2)}}\, dx\, dy = 16 \int_0^2 dx = 32 \text{ sq. units.}$$